THE DESIGN OF DIAGRAMS
FOR
ENGINEERING FORMULAS

THE DESIGN OF DIAGRAMS FOR ENGINEERING FORMULAS AND THE THEORY OF NOMOGRAPHY

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PREFACE

It is intended in this volume to present in a practical way the principles of the design of diagrams or nomograms for the solution of engineering and other formulas. The usefulness of a diagrammatic solution of a formula is becoming increasingly recognized and it is generally in proportion to the resistance of the formula to calculation and to the frequency of the application of the result sought. The aim of the present writing has been, therefore, not merely to give elementary methods of drawing simple diagrams but also to develop the grasp of the reader so that he will be able to analyze the more complex formulas of engineering practice.

The entire subject would only be handicapped by attempting to avoid the use of the third order determinants and consequently that notation is introduced in the third chapter and continued throughout the book. A sufficient treatment of determinants is given in Appendix A and is indispensable to those who are not familiar with that branch of college algebra.

The use of the projective transformation is mentioned, but the reader may proceed independently of that notion In Appendix B, however, is given a simple treatment of that subject sufficient to enable anyone who is interested to understand its application to the present theory.

By the determinant notation the identification of given formulas with known types is much helped although it is not completely furnished in all cases. It is hoped, however, that the necessary identification for these cases has been made much more complete by the introduction of an entire new class of diagrams or nomograms which it is proposed to call "Diagrams of Adjustment." These diagrams are new and are treated in the last chapter. All other diagrams of alignment are special cases of these more general types for they may naturally be regarded as diagrams of adjustment in which the adjustment reduces to zero.

The list of fifty-four illustrative examples is selected to avoid trivial instances. It is hoped that the careful presentation of the general theory of the introduction of scale factors and units of length into the diagram will enable the reader to produce designs that are practical. For this reason several difficult examples have been worked out in considerable detail.

The geometric theory governing the position of component elements such as curves, lines or points which constitute the permanent diagram must always be modified by the application of certain limits of accuracy and by a choice of the range of values of the variables for which the formula is to be used. The construction of a permanent diagram does not consist in the plotting of an indefinite number of results computed directly from the formula, but rather in a neat segregation of the several functions in the formula so that when certain corresponding scales are plotted and suitable simple geometrical constructions applied, a useful diagram results. The labor thus involved is usually slight compared to the resulting economy in the use of the formula for direct computation. Diagrammatic representation of a formula permits the immediate determination of the value of any variable and usually also permits the determination of the rate of variation of any variable with respect to another variable when such variations are not readily determined or observed by direct inspection of the formula.

The teaching of this subject of diagrammatic representation of formulas, or *Nomography*, at the Sheffield Scientific School for the past nineteen years has furnished opportunity to the authors to test its value as a supplementary course in applied mathematics and refined drafting, as well as in practice, and consequently all unnecessary theory has been sacrificed.

A comprehensive set of problems is given at the close of each chapter and many of them may easily be varied by the choice of method or of scale factors.

Acknowledgement is assuredly due to Professor M. d'Ocagne whose fundamental *Traité de Nomographie* doubtless awakened the present great interest in this fascinating subject and whose own sympathetic interest in an English exposition was expressed promptly.

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Washington, D. C.,
New Haven, Conn.,
May, 1923

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DESIGN OF DIAGRAMS FOR ENGINEERING FORMULAS

CHAPTER I

FUNCTION SCALES

1. The Function Scale.—In the construction of permanent diagrams for the numerical solution of formulas or equations it is constantly necessary to use a scale on which lengths are proportional to the values of a function of a single variable. The value of the variable is the important item so it is written beside the point determined by the corresponding value of the function. Thus, let it be required to construct the scale for the function $\sqrt{z}$ for values of z ranging from 0 to 5, then if unity is represented by 2 inches, the length of the scale will be

$$L = 2 \times \sqrt{5} = 2 \times 2.236 = 4.472 \text{ inches}$$

and its end will be marked 5. The number 4 will be written at the end of the segment $04 = 2\sqrt{4} = 4$ inches, 3 at the end of the segment $03 = 2\sqrt{3} = 3.464$ inches and so on. See Fig. 1.

a scale of the function $f(z)$. If it should happen that for necessary values of z the lengths OM are inconveniently large or small, these lengths may be modified by the introduction of a *scale factor* μ and laid off as

$$OM = \mu f(z)$$

Thus in the example worked out above where the linear unit is one inch, $\mu = 2$. The linear unit adopted on a drawing may also be called a *modulus*. Diagrams for engineering formulas involve more than one function scale in a figure and in such a diagram the modulus (*i.e.* unit of length) is usually adopted and suitable scale factors selected, as explained in Article 4. (See also Article 6, Chapter II.)

If $f(z)$ reduces to z itself, the resulting scale (Fig. 3) is the ordinary scale of the draftsman. This scale will be called the *ordinary scale*.

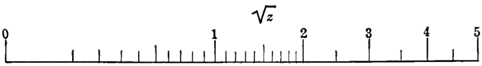


Fig. 1.

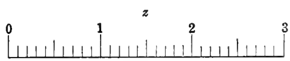


Fig. 3.

A very familiar example of a function scale is found in the common slide rule where the function is $\log z$, and the lengths are laid off proportional to the logarithms of the numbers z, as in Fig. 2.

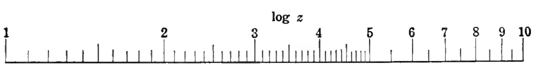


Fig. 2.

The notation $f(z)$, and more often simply f, will be used to denote any function of a single variable. Starting from an initial point O, if the successive lengths $OM = f(z)$ are laid off and the points M inscribed with the successive values of z, there results

2. Derivation of New Scales.—As there is often considerable computation necessary in the construction of a function scale it is desirable to make use of several graphical methods which help to establish the scales of new functions from scales already made.

(*a*) To establish the scale of

$$a + f(z)$$

from that of $f(z)$, where a is any constant, it is merely necessary to move the inscribed values of z forward or backward the distance a according as a is positive or negative.

(*b*) The next simple case is the change of the scale

factor of the function scale. Suppose the scale of $f(z)$ is established on the line MN (sometimes called the support of the scale), Fig. 4, with the scale factor

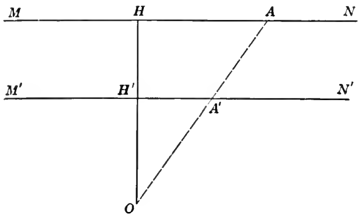


FIG. 4.

μ_1. The scale factor may be changed to μ_2 by simply drawing a line $M'N'$ parallel to MN and projecting the division points of the scale on MN to the line $M'N'$ from a point O such that

$$\frac{OH}{OH'} = \frac{\mu_1}{\mu_2}$$

Naturally this construction will also serve to set up with the same modulus, the scale of $af(z)$ from that of $f(z)$. A common case occurs when the scale log z is given and it is desired to obtain the scale of log z^a or a log z. In hydraulic formulas this opportunity is often presented.

(b^1) Other methods of handling the same problem are available especially where the supports for the two scales are not parallel. Suppose as before that the scale for $f(z)$ has been set up on the line MN, and the scale for $af(z)$ is desired on a line MN' with the same modulus. In Fig. 5 the two supports are shown

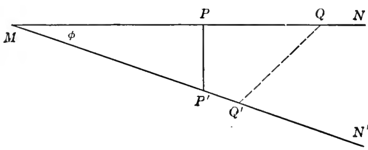


FIG. 5.

meeting at an angle ϕ. If QQ' is drawn at any convenient location such that

$$\frac{MQ}{MQ'} = \frac{1}{a}$$

it is simply necessary to project the points from the line MN to the line MN' by lines parallel to QQ'. In certain cases it may be desirable to project the points on MN parallel to PP' which is perpendicular to MN. In this case (assuming $a > 1$)

$$\cos \phi = \frac{1}{a}$$

Cases (a) and (b) combined furnish a method of constructing a scale for the function $a + bf(z)$.

The order of carrying out the work is immaterial.

(c) To set up the scale of $\frac{a}{f(z)}$ from the scale of $f(z)$ given on the line MN, Fig. 6, the procedure may be as follows:

Draw a circle with the center at M and with radius $\sqrt{a}$, and let $MP = f(z)$. Now if PT is tangent to the circle, and TP' is drawn perpendicular to MN, then $MP' \cdot MP = a$. Hence when the points P' have been marked with the same values of z as are found at the corresponding points P, the new scale is complete. If P is within the circle, then P' is without and is found by drawing TP perpendicular to MN and then drawing the tangent TP' to locate P'.

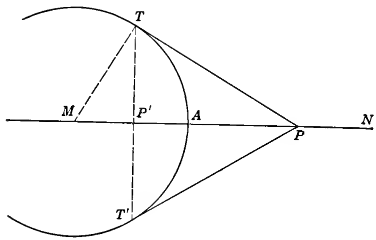


FIG. 6.

By making use of (a), (b) and (c) combined, the scale of

$$\frac{a}{bf(z) + c}$$

can also be obtained from that of $f(z)$.

(d) The functions in the preceding three cases are special cases of the more general case,

$$F(z) = \frac{af(z) + b}{cf(z) + d}$$

where $ad - bc$ is not equal to zero. To establish the scale for this function of z from that of $f(z)$ it is sufficient to project the division points of the scale of $f(z)$ from a point P, called a ray center, to a line $M'N'$ making an angle ϕ with the support MN, Fig. 7.

The most practical method of establishing the scale for $F(z)$ from that of $f(z)$ in this case is to compute the location of two points, z_1 and z_2, on the scale for $F(z)$

by substituting in the given function $F(z)$ two convenient values of z and plotting the resulting values of $F(z)$. Then the scale for $f(z)$ can be placed at an angle ϕ to the scale of $F(z)$ and corresponding values of z_1 and z_2 on $f(z)$ and $F(z)$ can be joined by rays the intersection of which determines the ray center P. The angle ϕ should be so chosen that the intersections of all rays with both scales are, as far as possible, not too oblique, and also so that the ray center P will not

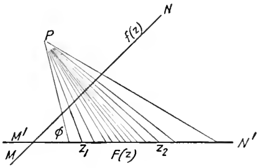


FIG. 7.

be located at too great a distance from the scales. Sometimes the ray center P may be located between the two scales. It is always well to check the position of P by a third ray through another pair of corresponding points on the two scales. Due to the above particular properties the scale for $F(z) = \dfrac{a\,f(z) + b}{c\,f(z) + d}$ is known as a projective scale.

(d^1) It is possible to compute the coordinates, m and n, of the ray center P from the values of a, b, c, and d in $F(z)$. The graphical method of the preceding paragraph is more direct but occasionally it may be desirable to check the location of the ray center P. If the supports of the two scales are used as coordinate axes (Fig. 8), the expressions for the oblique coordinates of P may be found. These will be

$$m = \frac{ad - bc}{c(cf(z_0) + d)} \qquad n = -\frac{cf(z_0) + d}{c}$$

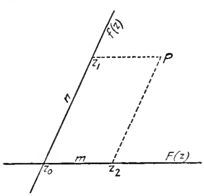


FIG. 8.

It is assumed that the scales intersect at corresponding points designated as z_0 on $f(z)$ and z_0 on $F(z)$.

$$\text{Since } F(z) = \frac{af(z) + b}{cf(z) + d}, \qquad f(z) = \frac{dF(z) - b}{-cF(z) + a}$$

Consider the projections obtained when a ray from z_1 on $f(z)$ parallel to the scale of $F(z)$ is drawn and another ray from z_2 on $F(z)$ parallel to the scale of $f(z)$ is drawn. In Fig. 8, z_1 must make $F(z_1) = \infty$.

$$\therefore\ f(z_1) = -\frac{d}{c}$$

Again $\qquad z_2$ must make $f(z_2) = \infty$

$$\therefore\ F(z_2) = \frac{a}{c}$$

Now $\qquad m = F(z_2) - F(z_0) = \dfrac{1}{c}(a - cF(z_0))$

But $\qquad F(z_0) = \dfrac{a\,f(z_0) + b}{c\,f(z_0) + d}$

Hence $\qquad m = \dfrac{ad - bc}{c(cf(z_0) + d)}$

Also $\qquad n = f(z_1) - f(z_0) = -\dfrac{cf(z_0) + d}{c}$

The coordinates m and n are thus determined by the value $z = z_0$ and this leaves the choice of the angle ϕ so that P can always be placed in the acute angle between the supports of the scales.

For example let

$$F(z) = \frac{4z + 7}{3z + 1}$$

If it is desired to plot the scale for this function by trial rays as described in section (d) above compute values of $F(z)$ for given values of z:

z	$F(z)$	Points
0	7	A
$\frac{1}{2}$	3.6	B
1	2.75	C
3	1.9	D
−2	0.2	E

Using a sheet of coordinate paper (Fig. 9) the scale for $F(z)$ is partially plotted along the horizontal axis from the origin O in points A, B, C, D and E. If at any point, say A, a scale for $f(z)$ which in this case is z itself (or the ordinary scale) is constructed, the rays to B, C, D and E through correspondingly numbered points on the two scales all intersect at P, the ray center. Then the remaining points on $F(z)$ could be graphically determined as fully as desired by projecting points from $f(z)$ to the support for $F(z)$. Here ϕ has been taken as 90°.

To illustrate the use of the method in section (d^1) it is noted that

$$a = 4 \qquad b = 7 \qquad c = 3 \qquad d = 1$$

In choosing a value for z_0 we are simply selecting a common point on the two scales $f(z)$ and $F(z)$. If

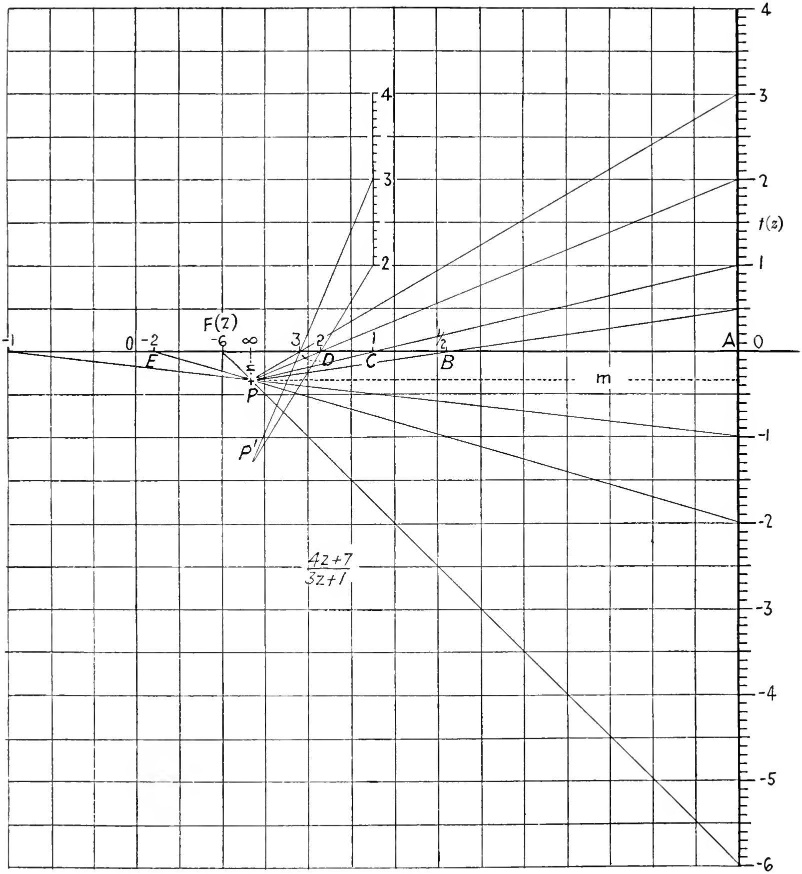


Fig. 9.

z_0 is chosen as zero, it means that at the point on $F(z)$ which is inscribed zero there is constructed a scale for $f(z)$ having its zero in coincidence with the zero of $F(z)$. Substituting in the expressions for m and n, noting that $f(z_0) = z_0 = 0$ there are obtained the values

$$m = -5.667$$
$$n = -0.333$$

Using the point of coincidence of the two scales as an origin (point A of Fig. 9) the ray center P may be located by measuring the values of m and n, as indicated in the figure. The scale for $F(z)$ can then be determined as fully as desired by projection from $f(z)$ as before.

If $f(z_0)$ had been chosen as unity in the preceding paragraph, the values of m and n would be

$$m = -1.417$$
$$n = -1.333$$

and the scale for $f(z)$ would have been erected at point C with its unity at C. The ray center would then be located at P' as shown and a new set of rays would determine the same points on $F(z)$ as before.

(*e*) The scale of

$$F(z) = f[\phi(z)]$$

may be obtained by using the scale of $f(z)$ as a measuring scale as follows. The quantity $\phi(z)$ plays the same part in the new scale as did z in the original scale of $f(z)$. Given z, the value of $\phi(z)$ can be computed, and regarding it as z the corresponding length can be picked out on the scale of $f(z)$. This length is the corresponding length on the new scale and is inscribed with the original value of z. Thus, for example, from the scale of log z there can be determined at once the scale of log tan z. There are in the trigonometric tables values of tan z for the values of z desired; then the points on the logarithmic scale are selected which are marked with these values of tan z. These lengths are then laid off on a new line as support and their end points are marked with the values of z (not tan z). (See Fig. 10 and Example 11.)

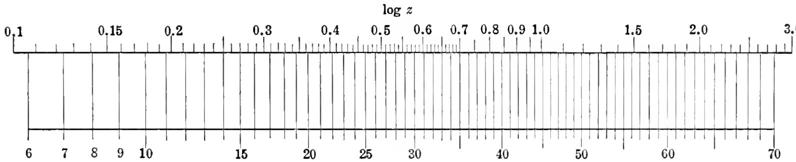


Fig. 10.

(*f*) The graph of a function may be used to advantage in setting up the scale of that function whenever the graph may be drawn mechanically either wholly or in part. Thus for example if

$$f(z) = z + \sin z$$

the graph OPA may be drawn, Fig. 11. If OM represents z, ON is of length $z + \sin z$ and if N is marked z, there is secured the desired function scale on OY. In the figure the curve OPA was drawn by adding the ordinates of the two curves B and C; B represents $f'(z) = z$ and C represents $f''(z) = \sin z$. Since the curve C may be obtained by the construction indicated in dotted lines on the right, the entire work of constructing the scale of $f(z) = z + \sin z$ can be done graphically. The graphical method becomes especially important when the analytical expression for a function is not known. This is usually the case when the graph of a function is obtained from experimental observations.

3. Equations in Two Variables.—Consider now a formula involving two variables,

$$z_2 = f(z_1)$$

If on one side of a line there is constructed the scale for the function $f(z_1)$ and on the other side the ordinary scale for z_2 with the same modulus and starting with corresponding values at the same point, then any pair of values which satisfy the above equation are found opposite each other on the two scales. Thus is realized by a diagram a numerical solution of the equation. To illustrate, if $f(2) = 5$, 2 is found on the function scale opposite 5 on the ordinary scale, and so on.

An obvious modification of this principle will permit the construction of a diagram yielding all solutions of an equation or formula of the form

$$F(z_2) = f(z_1) \tag{1}$$

Construct the scale for $F(z_2)$ on one side of a line as support and the scale for $f(z_1)$ on the other side and

read the value of z_2 opposite the value of z_1 corresponding. The scales must, of course, start at corresponding values a and b such that $F(a) = f(b)$, and have the same modulus. This case is useful when the formula is awkward to solve for either variable. The following simple examples will serve as illustrations:

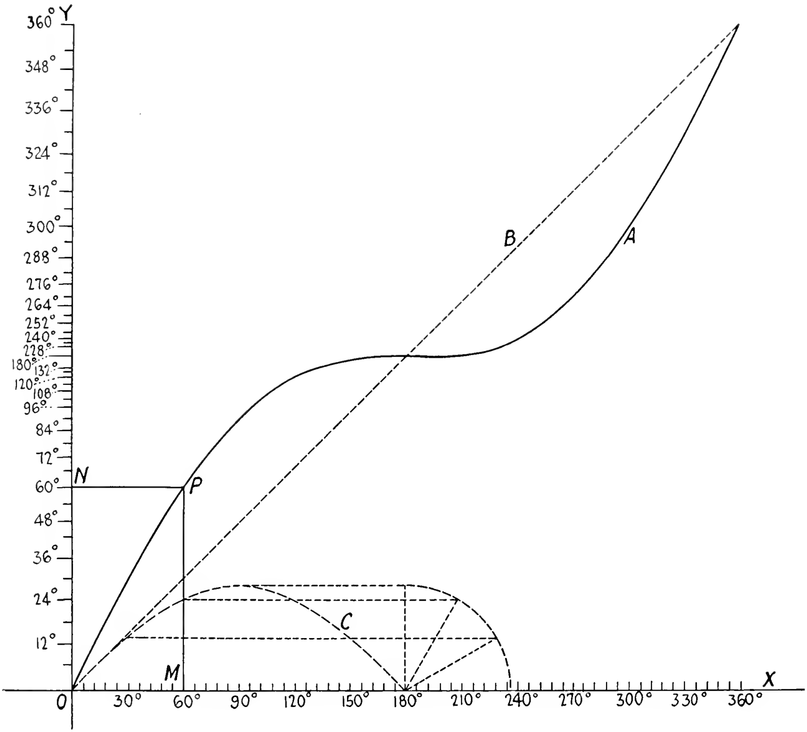


Fig. 11.

Example 1.—The formula for converting inches into hundredths of a foot and conversely may be written

$$100z_2 = 12z_1$$

where z_1 denotes hundredths of a foot and z_2 inches. Clearly all that is needed is to have a line divided on one side for inches and on the other side for hundredths of a foot as in Fig. 12.

Example 2.—The number of gallons z_2 in z_1 cubic feet may be written

$$z_2 = 7.481z_1$$

or $$\log z_2 = \log 7.481 + \log z_1$$

After laying out the logarithmic scale for z_2 on the upper side of the line in Fig. 13, construct the logarithmic scale for z_1 with its point marked 1 opposite 7.481 on the upper scale. Both scales have the same modulus and may be transferred from a slide rule or a logarithmic rule.

Example 3.—$\log z_1 = \frac{1}{2} \log z_2$ is the equation solved on the slide rule when square roots of z_2 are found.

Example 4.—The empirical formula

$$PV^{1.065} = 483$$

giving a relation between the pressure and volume of one pound of dry saturated steam, may be written,

$$\log P = \log 483 - 1.065 \log V.$$

Lay out a logarithmic scale of any convenient modulus on the lower side of the line as in Fig. 14, for the scale of P. At the point 483 on this scale is found the unity of the V scale, since when $V = 1, P = 483$. The scale factor of the logarithmic scale for V will be 1.065 and the scale will increase in the direction opposite to that of P. This scale may be constructed by the method of Article 2(b) or 2(b^1).

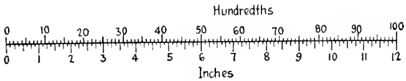


FIG. 12.

An equation of the form

$$z_1^a z_2^b = K$$

where K, a and b are constants, may be represented readily by the use of logarithmic scales. The exponents a and b really become the scale factors and one scale is translated a distance $\log K$.

Another method of treating an equation in two variables

$$z_2 = f(z_1)$$

is to make use of the ordinary cartesian graph. In Fig. 15 let C be the graph of the above equation referred to the axes OX and OY. For a given z_1 say OM, draw MP perpendicular to OX and from P drop a perpendicular PN to OY. Then ON is the desired value of z_2. Coordinate or cross-section paper would ordinarily be used for this type of diagram.

Diagrams representing equations in two variables are used more to supplement the usefulness of more complicated diagrams than to afford in themselves a means of solving equations in two variables. In later examples it will be found that many of the scales are graduated for two quantities, such as cubic feet per second and gallons per minute, on the same line. While only one of these quantities may appear in the formula for which the diagram is drawn, the addition of the other often increases the usefulness of the diagram.

4. Choice of Scale Factor.—The construction of the scale of a function with a suitable scale factor μ is an essential operation in the design of any permanent diagram for numerical solutions. The length L of the desired scale is limited by the size of the paper and must satisfy the equation

$$L = \mu[f(b) - f(a)]$$

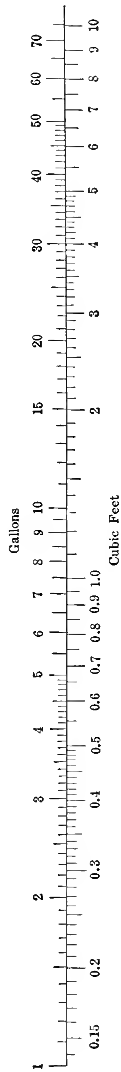


FIG. 13.

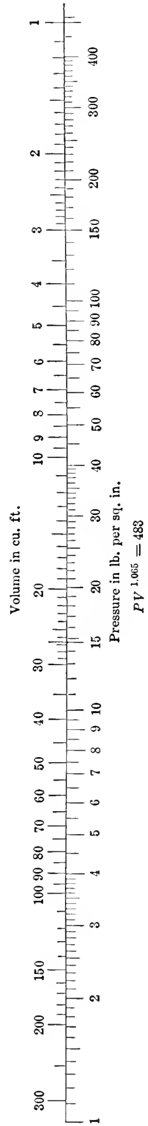


FIG. 14.

where a and b are the limiting values of the variable and μ is the scale factor to be selected. There is usually some choice of these limiting values a and b, and as more than one function scale is involved the relation of the various scales in the diagram must be carefully studied in advance. The use of the scale must be kept in view and the graduations arranged so that interpolations by eye will, when possible, yield one figure beyond the required accuracy. When some portion of a non-uniform scale is to be most frequently used that portion should be given the advantage of the larger graduations by the methods

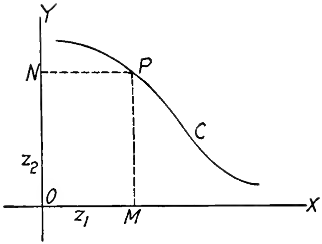


FIG. 15.

developed in Chapter III, as for example in the stadia formula

It is always desirable to check various points on a new scale by double calculations and by various known characteristics of the function such as the magnitude and uniformity of the rate of increase within a given interval of the variable. The accuracy of the finished diagram should also be checked by characteristics of the given formula and by various numerical examples.

Problem 1.—Construct a diagram showing the relation between kilowatts and horsepower.

Problem 2.—Construct a diagram showing the relation between circular pitch and diametral pitch of gear teeth.

Problem 3.—Construct the projective scale of the function

$$F(z) = \frac{1.7 \log z + 6.5}{2.4 - 0.84 \log z}$$

from a logarithmic scale.

Problem 4.—Construct a scale for values of ϕ from 0° to 135° for the function $\tan\left[\frac{\pi}{4} + \phi\right]$.

Problem 5.—Plot the function scale for $\frac{1}{z}$ between the limits $z = 2$ and $z = 50$ upon a line 12 inches long.

Problem 6.—Plot a scale for the function

$$\frac{1 + \sin z}{1 - \sin z}$$

for values of z from 0° to 30°.

Problem 7.—Plot a scale for the function $\frac{H^{3/2}}{H^{3/2} + 2}$ on the line OX.

Problem 8.—Establish the function scales $\frac{2t + 3}{t - 1}$ and $\frac{3t + 1}{t - 1}$ on the X and Y axes respectively and show that corresponding values of t determine values of the coordinates that locate points on a straight line. What is the equation of this line in cartesian coordinates?

Problem 9.—Plot the scale for $0.38V^{1.86}$ starting from a definite point O on the line OX.

Problem 10.—Construct a scale for the law

$$v = \sqrt{2gh}$$

between the limits $h = 1$ and $h = 100$.

Problem 11.—Construct a scale for $\left(0.405 + \frac{0.0098}{h}\right)$ between the limits $h = 0.2$ and $h = 1.4$.

Problem 12.—Construct a diagram for the velocity v due to an adiabatic heat drop ΔH for steam from the expression $v = 223.8\sqrt{\Delta H}$.

Problem 13.—Construct a diagram for $P^{.9406}V = 327.7$.

CHAPTER II

ELEMENTARY DIAGRAMS

5. Simple or Elementary Diagrams.—An equation or formula involving three variables z_1, z_2, and z_3 may be denoted by

$$f(z_1z_2z_3) = 0$$

or more briefly by

$$f_{123} = 0 \tag{2}$$

One of the main objects of this volume is to develop the construction of a permanent diagram for solving Equation (2). Such a diagram should determine any third variable when two are given and it is frequently called a *nomogram.* A natural method would be to let z_1 and z_2 represent independent variable coordinates x and y and plot the family of curves

$$f(xyz_3) = 0$$

In such a diagram the parameter z_3 which varies from curve to curve should be the variable whose values from the nature of the given problem increase by fixed intervals.

Example 5.—To illustrate, consider the formula

$$S = \frac{P - D}{P}$$

which gives the proportion of strength S remaining in a plate at a riveted joint, where P is the pitch and D the diameter of the rivet holes, both in inches. In this formula P and S have almost any values (within certain limits) while D usually varies by sixteenths of an inch. Take then

$$P = x \quad \text{and} \quad S = y$$

leaving D as the parameter of the system which will require the least number of curves. There results then the simple curve equation

$$y = \frac{x - D}{x}$$

with values of D ranging from $\frac{1}{2}$ to $1\frac{1}{2}$ inches by eighths or by sixteenths if desired. Substituting $D = \frac{1}{2}$ in the equation of the curve family gives

$$y = \frac{x - \frac{1}{2}}{x}$$

Plot this curve and mark it $D = \frac{1}{2}$. One could then proceed to plot each of the curves for values of D equal to $\frac{5}{8}$, $\frac{3}{4}$, $1\frac{1}{2}$, etc., and thus obtain the system of curves for D as shown in Fig. 16. Since, however, for any two successive values D' and D'' of D the corresponding ordinates y_1 and y_2 for a given abscissa x_1 are

$$y_1 = \frac{x_1 - D'}{x_1} \quad \text{and} \quad y_2 = \frac{x_1 - D''}{x_1}$$

it follows that

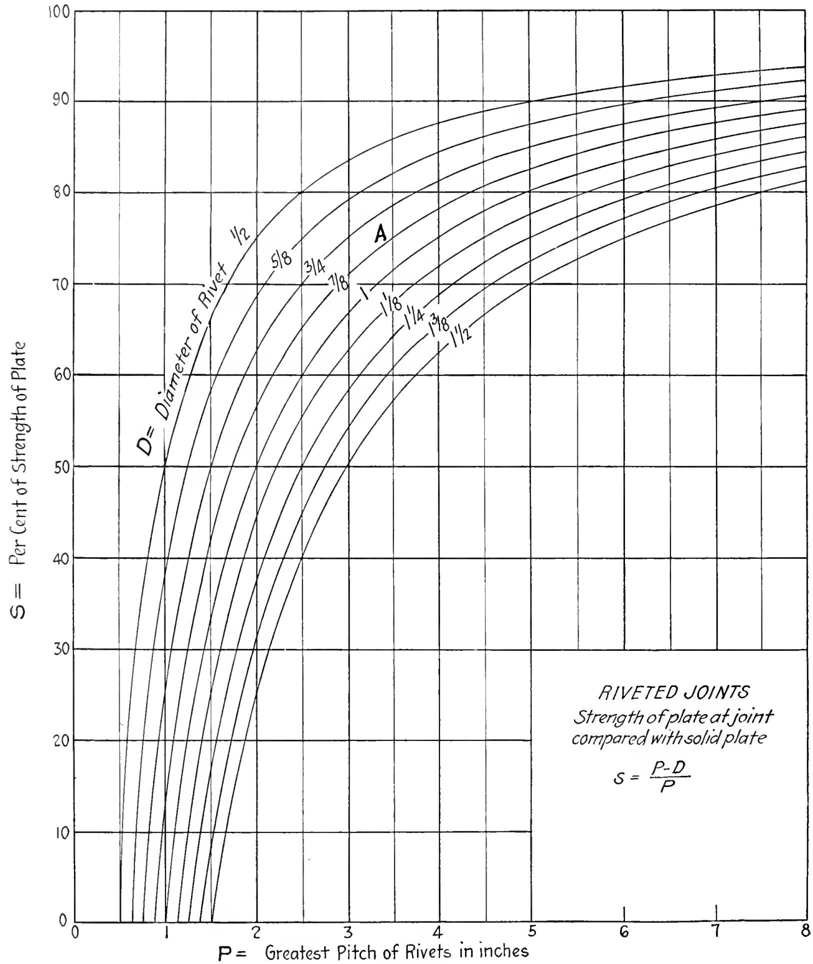
$$y_1 - y_2 = \frac{x_1 - D'}{x_1} - \frac{x_1 - D''}{x_1} = \frac{D'' - D'}{x_1}$$

that is to say for the same abscissa, the increments of any ordinate for equal increments of D are equal. It is necessary then to plot only the extreme curves for $D = \frac{1}{2}$ and $D = 1\frac{1}{2}$ and divide the portion of each vertical line which is included between them, into the same number of equal parts. In Fig. 16 the ordinates were divided into eight equal parts.

To find S from the diagram when P and D are given: Find the intersection of the ordinate at the point on the X axis corresponding to the given value of P with the curve marked with the given value of D and then read the required value of S horizontally on the Y axis. For example, if $P = 3\frac{1}{2}$ inches and $D = \frac{7}{8}$ inch, find the ordinate $3\frac{1}{2}$ and the curve $D = \frac{7}{8}$ intersecting at the point A, which is opposite the value 75 on the Y axis. The figure may be entered with any two variables and the third found. Suppose, for example, it is required to determine what pitch will be required with $\frac{3}{4}$ inch rivets to give an efficiency of 80 per cent. From the point 80 on the Y axis run horizontally to the right as far as the curve $D = \frac{3}{4}$, then run vertically to the X axis and the required value of P is 3.75 inches.

6. Scale Factors.—The scale factors used when the ordinary scales for z_1 and z_2 are constructed on the axes of coordinates need not be the same. It is usual to have

$$x = \mu_1 z_1 \quad \text{and} \quad y = \mu_2 z_2$$



S = Per Cent of Strength of Plate
P = Greatest Pitch of Rivets in inches
D = Diameter of Rivet
1/2
5/8
3/4
7/8
1
1 1/8
1 1/4
1 3/8
1 1/2
A
RIVETED JOINTS
Strength of plate at joint compared with solid plate
S = (P-D)/P

Fig. 16.

Then for the equation $f_{123} = 0$ the curve system becomes

$$f\left[\frac{x}{\mu_1}, \frac{y}{\mu_2}, z_3\right] = 0$$

This equation is the one from which the family of curves z_3 is plotted to ordinary and equal scales on both axes. It is necessary to introduce such scale factors when one of the independent variables, say z_1, varies through a greater range than the other z_2.

Example 6.—In Fig. 17 there is shown a diagram for Francis' formula for the discharge of water over a weir without end contractions,

$$q = 3.33BH^{3/2}$$

where H is the head of the water on the crest of the weir in feet, B the width of the weir in feet, and q the discharge in cubic feet per second. Of these three variables the breadth B is the one which is most likely to be expressed in even numbers, so it is chosen for the parameter of the system of necessary curves. The head H is usually less than two feet while with $B = 10$ the discharge q runs up to 94.2 if $H = 2$. So the values of q run through a range of numbers about 50 times as great as the corresponding values of H. Accordingly it will be desirable to plot the scale for H with a scale factor which is about 50 times that used for the scale of q.

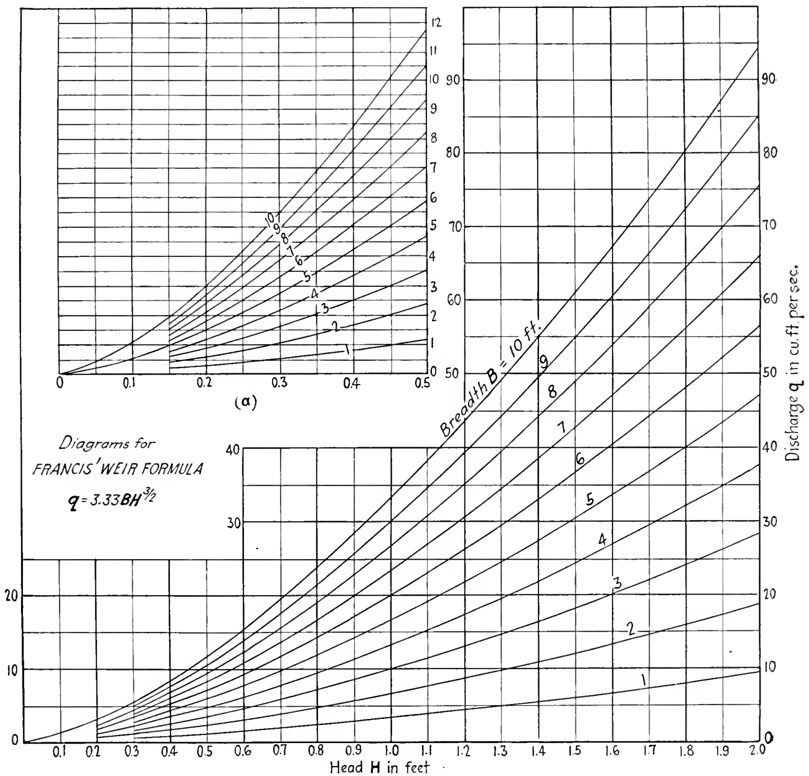


Fig. 17.

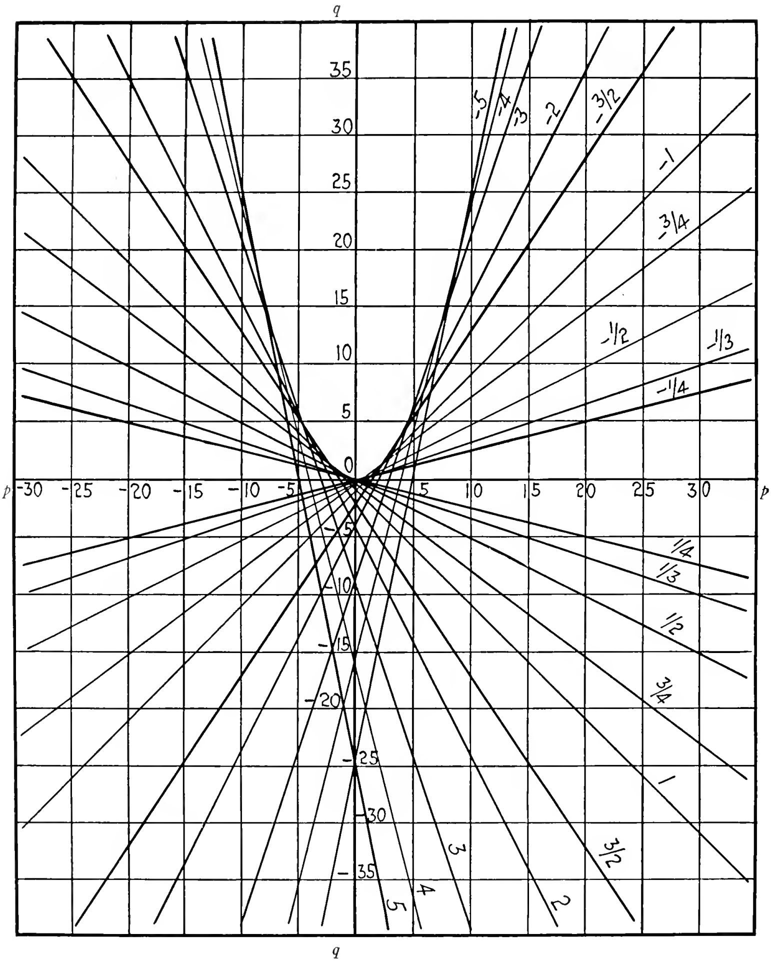


Diagram for the quadratic $z^2 + pz + q = O$

Fig. 18.

Let

$$x = \mu_1 H \quad \text{and} \quad y = \mu_2 q$$

then

$$\frac{y}{\mu_2} = 3.33B\left(\frac{x}{\mu_1}\right)^{3/2}$$

is the equation of the system of curves for B. In Fig. 17 μ_1 was taken as 5 and μ_2 as 0.1 so that

$$y = 0.2978x^{3/2}B$$

is the equation of the curve system referred to natural scales on the coordinate axes. Since each value of $0.2978x^{3/2}$ is multiplied by B it is necessary to plot only the curve for $B = 10$ and divide each of its ordinates into 10 equal parts to obtain the entire system of curves for B. In Fig. 17*a* the curves near the origin are shown drawn to a larger scale.

7. Simple Straight Line Diagrams.—The labor of constructing diagrams such as were given above is considerable unless the family of curves is easily plotted. The curves can be made straight lines whenever Equation (2) has the form

$$z_1 f(z_3) + z_2 g(z_3) + h(z_3) = 0$$

or more briefly

$$z_1 f_3 + z_2 g_3 + h_3 = 0 \tag{3}$$

where f_3, g_3 and h_3 are any functions of z_3, may or may not be alike and frequently reduce to constants including zero. The use of the same letter to denote functions of different variables in what follows will not necessarily mean that the functions are the same although such may sometimes be the case. In general, for example, $f(z_1)$ or f_1 will not denote the same functions as $f(z_3)$, etc.

Whenever Equation (2) has a form which may be reduced to the above form (3) by suitable transformations, set

$$x = \mu_1 z_1 \quad \text{and} \quad y = \mu_2 z_2$$

and Equation (3) becomes

$$x\mu_2 f_3 + y\mu_1 g_3 + \mu_1\mu_2 h_3 = 0$$

which determines a family of straight lines marked with corresponding values of z_3. Equations in three variables such as Equation (2) occur very frequently in engineering practice and are of particular interest here.

Example 7.—The general quadratic equation

$$z^2 + pz + q = 0$$

if $x = p$ and $y = q$ becomes

$$z^2 + xz + y = 0$$

This is a straight line system and the original Equation is of the type (3) where

$$\mu_1 = \mu_2 = 1$$

and

$$f_3 = z, \qquad g_3 = 1, \qquad h_3 = z^2$$

When as many lines of the system have been drawn as the diagram will comfortably admit at suitable intervals of z it is seen that for moderate values of the coefficients p and q it is possible to solve any quadratic by reading the roots written on the lines passing through the corresponding intersection point of the lines $x = p$ and $y = q$. It will be necessary to interpolate for all the quantities p, q and z. See Fig. 18.

8. Anamorphosis.—It is possible in a large class of equations which do not fall under the type of Equation (3) to reduce the needed family of curves to straight lines. It will first be shown how this may be done graphically with a single curve and then the method will be extended to apply to a family of curves.

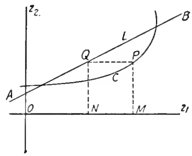


FIG. 19.

Suppose there is given in Fig. 19 a single curve C corresponding to some particular value of the quantity z_3 in the equation

$$f(z_1z_2z_3) = 0$$

This curve may be changed to a straight line L which will serve equally well to determine either of the corresponding quantities z_1 and z_2 as follows. Draw any oblique line AB and let every point P of the curve C be projected horizontally into a corresponding point Q upon the line L. Now inscribe N, the foot of the ordinate of Q, with the value of z_1 which is found at M on the X axis. After a sufficient number of points have been treated in this way the curve C may be erased, also the old scale of z_1 and then the diagram serves to determine the corresponding values of z_1 and z_2 for the value of z_3 originally used. This process was called by Lalanne "Anamorphosis."[1] What has been done changes the scale on OX from the ordinary scale for z_1 to a certain function scale. To see this it is only necessary to notice that the length ON is always a function of the length OM.

A logical extension of the above principle to all the curves z_3 of a given family is desirable. For this purpose it will be necessary from the given equation

$$f(z_1z_2z_3) = 0$$

to select a function x of z_1 such that when $y = \mu_2 z_2$ the entire family of curves corresponding to values

[1] L. LALANNE, *Annales des Ponts et Chaussées*, 1846.

of z_3 shall be straight lines. That is, it is necessary to change the original equation

$$f(z_1 z_2 z_3) = 0$$

by virtue of the relations,

$$x = \mu_1 f(z_1) \qquad y = \mu_2 z_2$$

into a linear equation in x and y. A necessary and sufficient condition is that the original equation

$$f(z_1 z_2 z_3) = 0$$

may be reduced to the form

$$f(z_1)f_3 + z_2 g_3 + h_3 = 0 \tag{4}$$

For in this Equation (4) if z_1 and z_2 are eliminated there results

$$x\mu_2 f_3 + y\mu_1 g_3 + \mu_1\mu_2 h_3 = 0$$

which is the equation of a family of straight lines to be inscribed with values of z_3.

Equation (4) is of the form that will yield straight lines when a function scale is used on the X axis only. If, however, the ordinates also are made to depend not simply on z_2 but on a function of z_2, as $f(z_2)$, there results a method of treating equations of greater generality. Set therefore

$$x = \mu_1 f_1 \quad \text{and} \quad y = \mu_2 f_2$$

then when the Equation (2), $f_{123} = 0$, has the form

$$f(z_1)f_3 + f(z_2)g_3 + h_3 = 0 \tag{5}$$

it will yield a system of straight lines for the values of z_3, by virtue of these relations.

This is the principle underlying the use of "logarithmic cross-section paper" for plotting an equation in two variables. This paper is a cross-section paper ruled with logarithmic scales on the axes instead of with the ordinary scales. Any equation in two variables which has the form

$$z_1{}^a z_2{}^b = K,$$

for example, where a, b and K are constants, may immediately be given the form

$$a \log z_1 + b \log z_2 - \log K = 0$$

by taking the logarithm of both sides. The resulting equation has the form (5). When therefore

$$x = \log z_1 \qquad y = \log z_2$$

the above equation reduces immediately to

$$ax + by - \log K = 0$$

which is a straight line equation for the ordinary cross-section paper. Or in other words, if corresponding values of z_1 and z_2 determined from the original equation are plotted directly on the logarithmic cross-section paper, the resulting coordinates are proportional to the corresponding logarithms and the graph is a straight line.

The exponents a and b determine the slope of the resulting straight line; *i.e.* $-\frac{a}{b}$.

This principle when used inversely is of great value in determining the unknown exponents for an empirical formula when a sufficient number of points are plotted on the logarithmic cross-section paper from actual observation and are found to determine closely a straight line.

Equation (5) is a very general type equation and includes a large number of formulas of engineering. Such formulas will frequently require algebraic and sometimes logarithmic transformations in their form before they can be identified with the type by inspection. It will be seen that the corresponding diagrams consist essentially of three systems of straight lines and that two of these systems are parallel to the axes, determined by function scales on the axes.

The foregoing Equation (5) is not the most general equation in three variables whose diagram can be constructed by three straight line systems provided no restriction is placed on the nature of the systems. Such an equation is best expressed in determinant form but can, however, be treated by much more elegant methods than those of the present chapter.

Example 8.—The "external" or distance from the intersection of two tangents to the curved line, in highway or railroad surveying is given by the formula

$$b = T \tan \frac{I}{4}$$

where T is the length of the tangent and I the acute angle of intersection. In the field it is often desired, before finally determining either T or b for a given angle, to try several pairs of values, and the diagram given in Fig. 20 is convenient.

The formula is in the form of Equation (4) where

$$\tan\frac{I}{4} = f(z_1),\ b = z_2,\ T = f_3,\ g_3 = -1,\ h_3 = 0$$

so that if

$$x = \mu_1 \tan\frac{I}{4} \text{ and } y = \mu_2 b$$

there results the radial line system

$$\frac{y}{\mu_2} = \frac{x}{\mu_1} T$$

the value of $\tan\frac{I}{4}$ for $I = 30°$ is 0.1317 and if the limit of b is taken as 18 feet the diagram of Fig. 20 can be drawn with $\mu_1 = 0.3$ and $\mu_2 = 60$.

Example 9.—The mean pressure P_m of steam expanded from an initial pressure P_1 according to the law $PV =$ constant, is given by the formula

$$P_m = P_1 \frac{1 + \log_e R}{R}$$

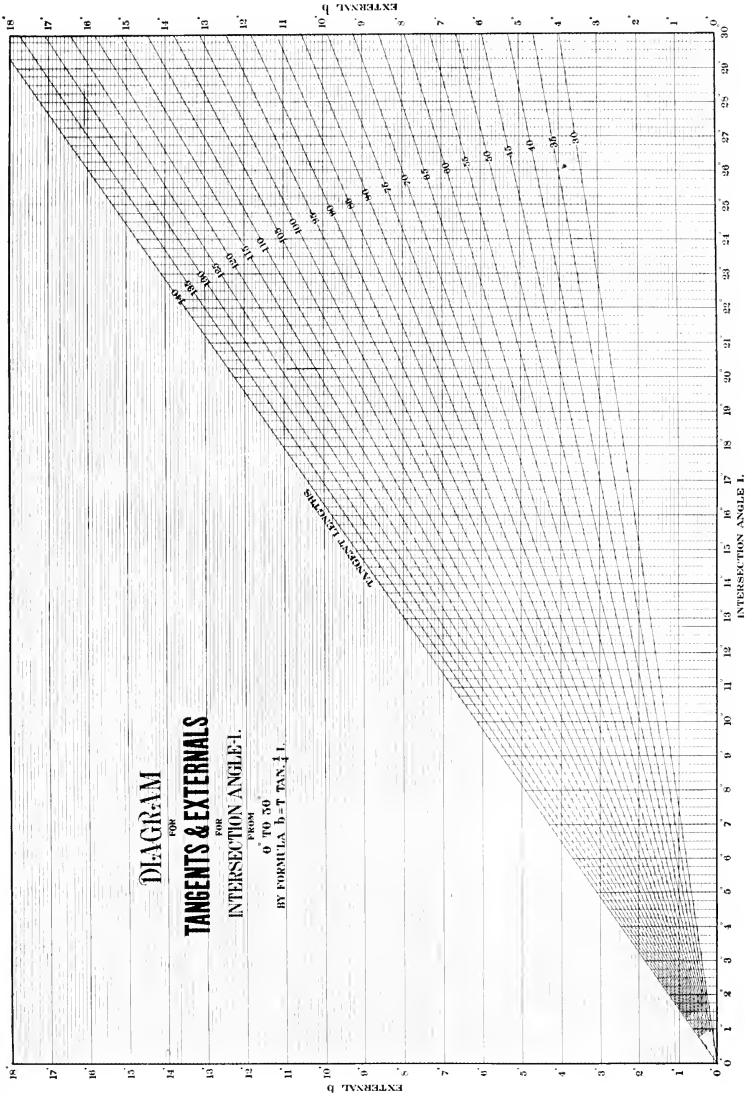


DIAGRAM
FOR
TANGENTS & EXTERNALS
FOR
INTERSECTION ANGLE-I.
FROM
0° TO 30°
BY FORMULA b=T TAN.¼ I.
TANGENT LENGTHS
EXTERNAL b
EXTERNAL b
INTERSECTION ANGLE I.

Fig. 20.

if measured above a back pressure of absolute zero. R is the ratio of expansion.

If
$$\frac{1 + \log_e R}{R}$$
is taken as f_1 and P_m as f_2 then $f_3 = P_1$, $g_3 = -1$ and $h_3 = 0$ showing that the above equation is in the form of (5)

$$f_2 - f_3 f_1 = 0$$

Accordingly let

$$x = \mu_1 \frac{1 + \log_e R}{R}$$

and
$$y = \mu_2 P_m$$

so that there results a family of radial straight lines

$$\frac{y}{\mu_2} = P_1 \frac{x}{\mu_1}$$

(See Fig. 21.)

Since all the lines P_1 pass through the origin it is necessary to locate but one point on each line to draw the system. Such points are very simply determined by the intersections of the radial lines with the line $x = 1$ parallel to the Y axis. In general when a system of radial lines $y = mx$ is to be plotted, set $x = 1$ so that $y = m$. In the present case the scale determined on the line $x = 1$ is an ordinary scale whose scale factor is $\frac{\mu_2}{\mu_1}$. Of course beyond the limits of the paper the radial lines cannot intersect the line $x = 1$ and if it is necessary to draw additional lines they may be determined by their intersections with a line parallel to the Y axis at any convenient distance.

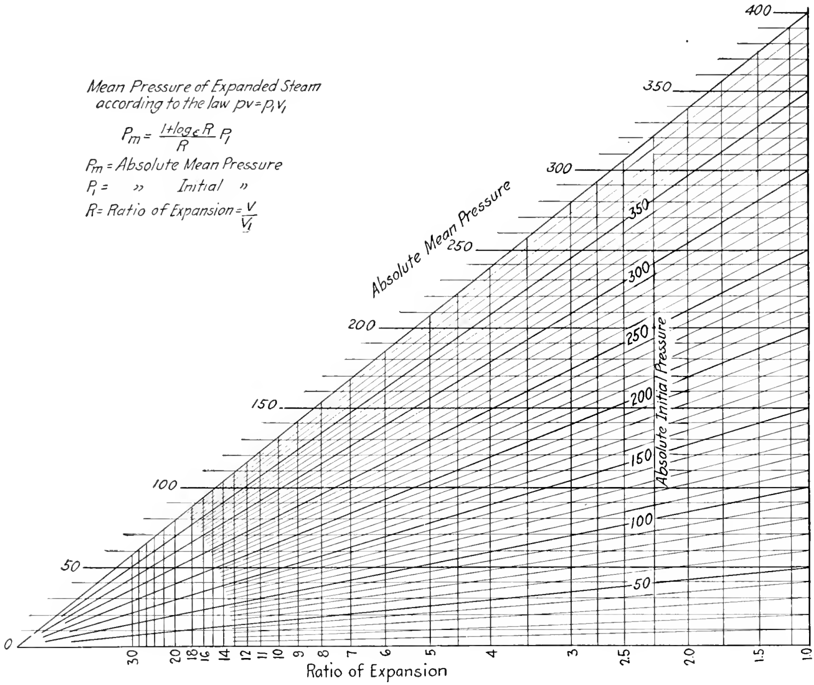


FIG. 21.

Example 10.—The formula $S = \frac{P - D}{P}$ of Example 5 may be written

$$D = P(1 - S).$$

If $f_1 = (1 - S)$, $f_2 = D$, $f_3 = P$, $g_3 = -1$, $h_3 = 0$ it is in the form of Equation (5)

$$f_1 f_3 - f_2 = 0$$

Accordingly let $x = \mu_1(1 - S)$

$$y = \mu_2 D$$

giving $$\frac{y}{\mu_2} = P\frac{x}{\mu_1}$$

as shown in Fig. 22.

it becomes

$$ny - x = 0 \text{ as shown in Fig. 23.}$$

The scales on the axes are readily plotted by the method of Article 2 (*e*); *i.e.*, for a given α or β look up the value of the natural tangent, add one, and find the resulting quantity on a logarithmic scale, inscribing the point with the value of α or β used.

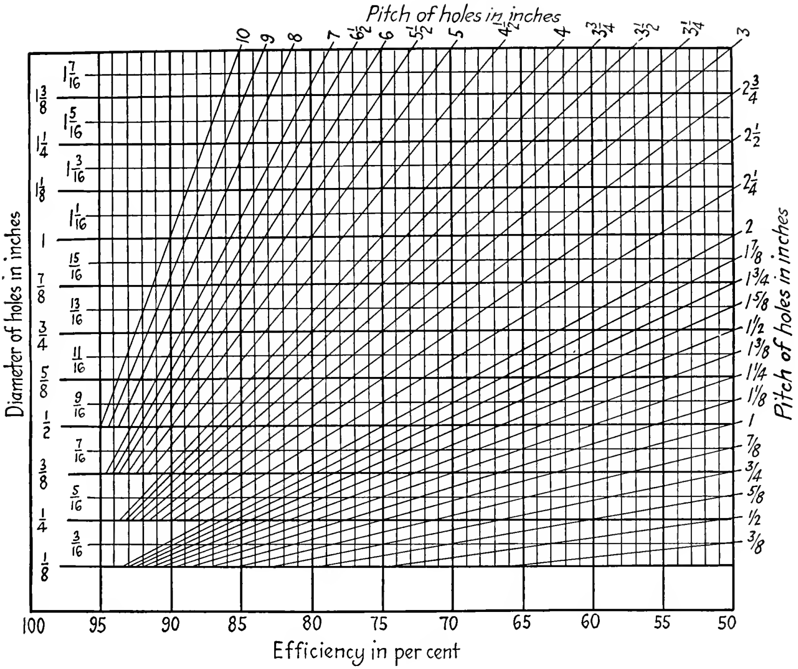


Fig. 22.

It will be noticed that the graduations of the S scale on the X axis increase toward the origin, since the function is $(1 - S)$.

Example 11.—The expression $(\tan \alpha + 1)^n = (\tan \beta + 1)$ is useful in plotting exponential curves of the type PV^n = constant, in thermodynamics by Brauer's method. If written

$$n \log (\tan \alpha + 1) - \log (\tan \beta + 1) = 0$$

with

$$x = \log (\tan \beta + 1)$$
$$y = \log (\tan \alpha + 1)$$

Example 12.—The formula for the diameter (d) of a shaft to transmit a given horsepower (h.p.) at a given speed (r.p.m.) is of the form

$$d = c\sqrt{\frac{\text{h.p.}}{\text{r.p.m.}}}$$

If the allowable stress for a steel shaft is taken as 13,500, the constant c has the value 2.87; hence

$$d^3 = (2.87)^3\text{h.p.}\frac{1}{\text{r.p.m.}}$$

If a reciprocal scale is used on the X axis and a cubic

scale is used on the Y axis a family of straight lines for values of h.p. results.

Let $$x = \frac{\mu_1}{\text{r.p.m.}}$$

and $$y = \mu_2 d^3$$

Example 13.—The approximate formula for the area A of the segment whose height is H of a circle of radius R is

$$A = \frac{4H^2}{3}\sqrt{\frac{2R}{H} - 0.608}$$

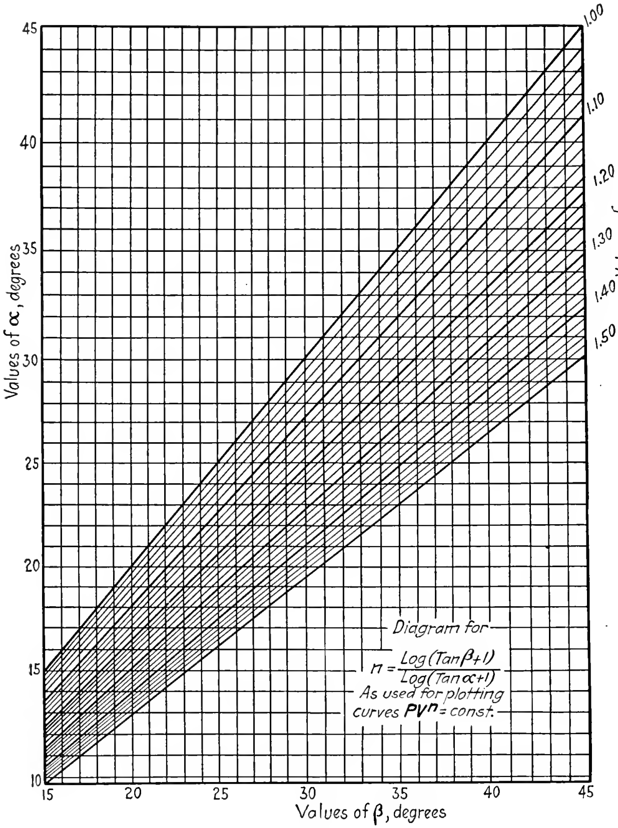


FIG. 23.

so that $$y = \frac{\mu_2}{\mu_1}(2.87)^3 \text{ h.p.}x$$

as shown in Fig. 24. A second set of underscored graduations for d and h.p. have been added, covering a larger range of numbers.

or $$9A^2 - 32RH^3 + 9.728H^4 = 0$$

This equation is in the form (5) with $f_1 = A^2$ and $f_2 = R$ so that if $x = A^2$ and $y = R$ a family of straight lines for H

$$9x - 32H^3y + 9.728H^4 = 0$$

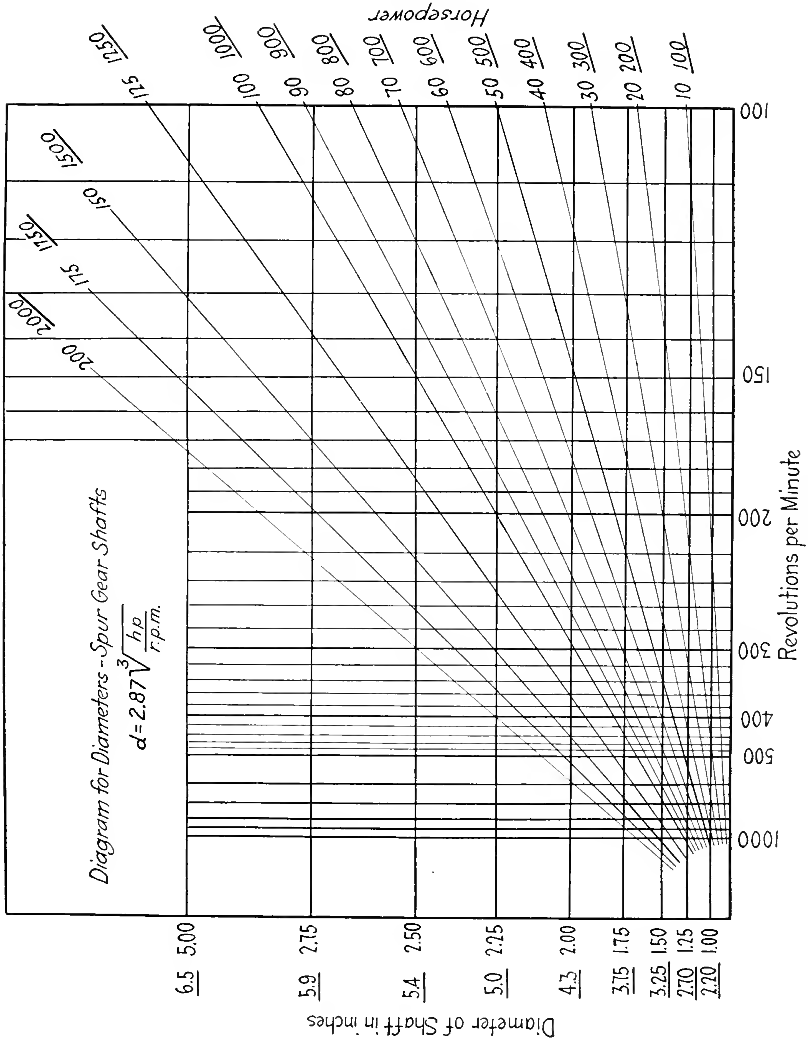


Diagram for Diameters - Spur Gear Shafts
d = 2.87 ∛(h.p / r.p.m.)
Horsepower
10 100
20 200
30 300
40 400
50 500
60 600
70 700
80 800
90 900
100 1000
125 1250
150 1500
175 1750
200 2000
Revolutions per Minute
100
150
200
300
400
500
1000
Diameter of Shaft in inches
5.00 6.5
2.75 5.9
2.50 5.4
2.25 5.0
2.00 4.3
1.75 3.75
1.50 3.25
1.25 2.70
1.00 2.20

Fig. 24.

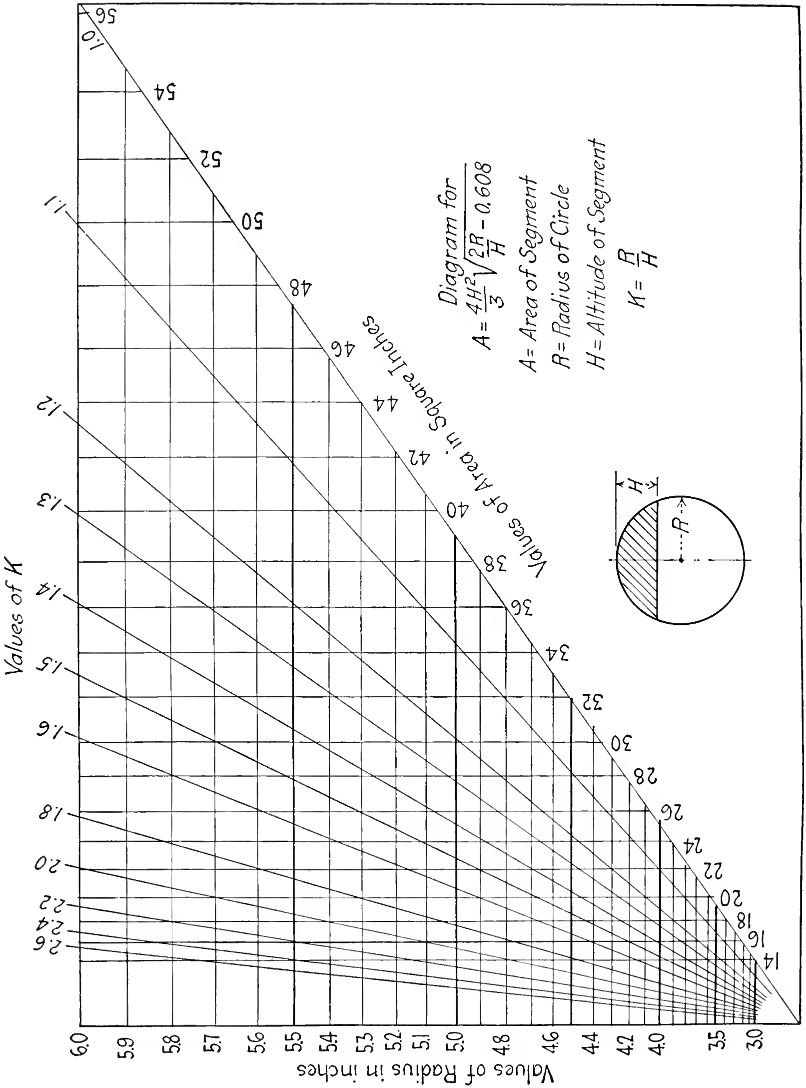


Diagram for
A = 4H²/3 √(2R/H − 0.608)
A = Area of Segment
R = Radius of Circle
H = Altitude of Segment
K = R/H
Values of K
Values of Area in Square Inches
Values of Radius in inches
H
R

FIG. 25.

would result. This equation is difficult to plot and the lines are poorly located for accurate reading.

If the ratio $\frac{R}{H} = K$ is used the equation becomes

$$A = \frac{4}{3} R^2 \frac{\sqrt{2K - 0.608}}{K^2}$$

Then with $x = \mu_1 f_1 = \mu_1 A^2$ and $y = \mu_2 f_2 = \mu_2 R^4$ there results a radial system $x = \frac{\mu_1}{\mu_2} \frac{16}{9} y \frac{2K - 0.608}{K^4}$ as shown in Fig. 25.

9. Special Form of Equation.—The case where Equation (5) has the simple form

$$f_1 + f_2 + f_3 = 0 \tag{6}$$

is of special importance. It gives rise to a system of parallel straight lines, since if $x = \mu_1 f_1$ and $y = \mu_2 f_2$, the equation becomes

$$\mu_2 x + \mu_1 y + \mu_1 \mu_2 f_3 = 0$$

This system of lines may be dispensed with if their common normal through 0 is drawn and on it the function scale for z_3 established. The function scales for z_1 and z_2 must be constructed on the X and Y axes respectively as before. The diagram then consists essentially of three function scales whose supports intersect at 0. It is read by finding the unknown value of z where a line through the intersection of the two perpendiculars at the given values of z on their respective axes meets perpendicularly the scale of the unknown z. See Fig. 26.

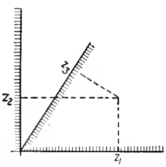


FIG. 26.

Since the three lines necessarily perpendicular to the respective scales meet at constant angles they may be scratched on a transparent sheet which when properly oriented on the drawing will enable the unknown values to be read rapidly. For ordinary work, however, a diagram having the cardinal values of all three straight line systems drawn in is found to be the best arrangement.

Example 14.—The formula for the weir discharge used in Example 6

$$q = 3.33 B H^{3/2}$$

may be brought into the form (6) by taking the logarithm of both sides. There results

$$\log q - \log 3.33 - \log B - \tfrac{3}{2} \log H = 0$$

Here if $\log q = f_2$ and $\log H = f_1$ it is seen that

$$f_3 = -\log 3.33 - \log B$$

Set $x = \mu_1 \log H$ and $y = \mu_2 \log q$. Then the equation of the parallel lines for B is

$$\frac{y}{\mu_2} - \frac{3}{2} \frac{x}{\mu_1} - \log 3.33 - \log B = 0$$

These lines may best be drawn if the common normal to the system is first drawn and numbered with the values of B at the points of intersection with the parallel lines. To do this it is necessary to determine

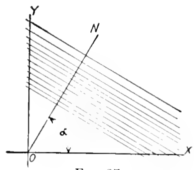


FIG. 27.

the angle α of Fig. 27 (above) and the corresponding function scale on the normal. The angle in the present example is 126° 52′ 12″ and the lines B intersect the normal at distances from the origin determined by the function

$$\frac{4}{5} \mu_1 [\log 3.33 + \log B]$$

The completed diagram is shown in Fig. 28.

In general when an equation is of the form (6) and the resulting system of lines for z_3 is given by the Equation

$$\mu_2 x + \mu_1 y + \mu_1 \mu_2 f_3 = 0$$

this last equation may be put into the normal form

$$x \cos \alpha + y \sin \alpha - p = 0$$

where

$$\cos \alpha = \frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} \text{ and } \sin \alpha = \frac{\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}}$$

and where the scale on the normal is determined by the function f_3 with the scale factor

$$\frac{-\mu_1 \mu_2}{\sqrt{\mu_1^2 + \mu_2^2}}$$

Example 15.—The formula $H = 0.38 \frac{V^{1.86}}{d^{1.25}}$ gives the friction head H in feet per 1,000 feet of water flowing in a pipe of diameter d with a velocity of V feet per second. In logarithmic form the equation is

$$\log H + 1.25 \log d - \log 0.38 - 1.86 \log V = 0$$

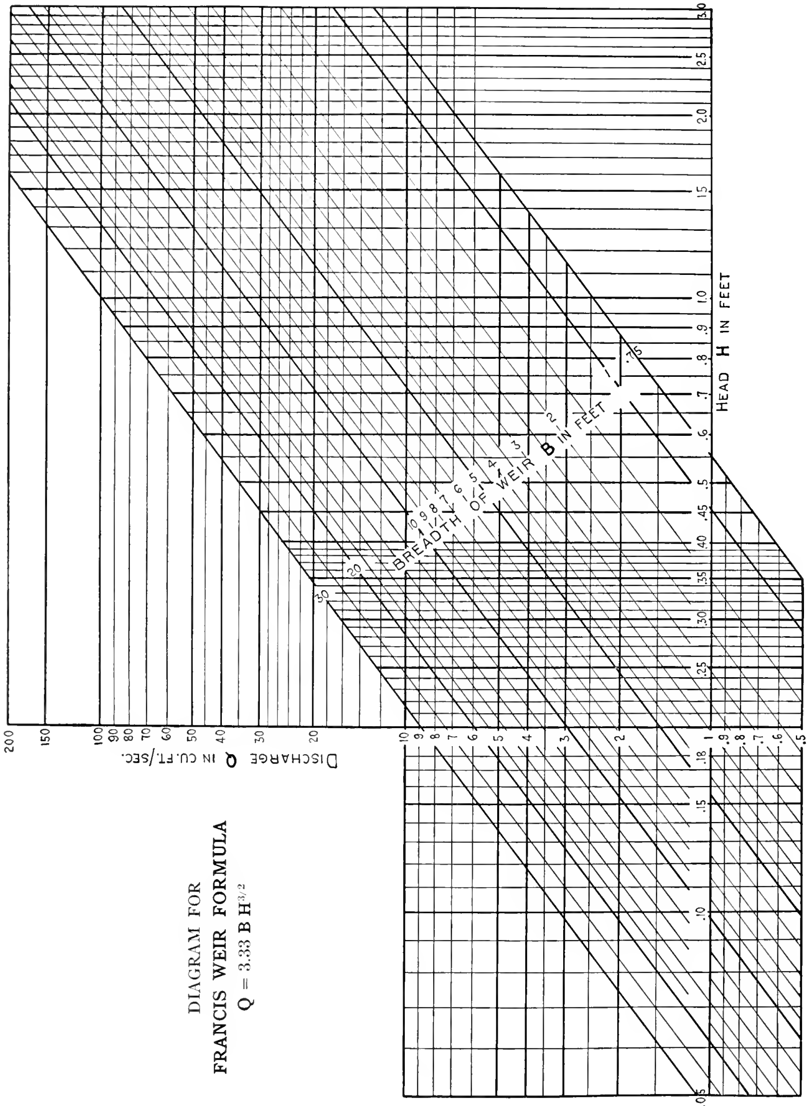


DIAGRAM FOR
FRANCIS WEIR FORMULA
$Q = 3.33\ B\ H^{3/2}$

Fig. 28.

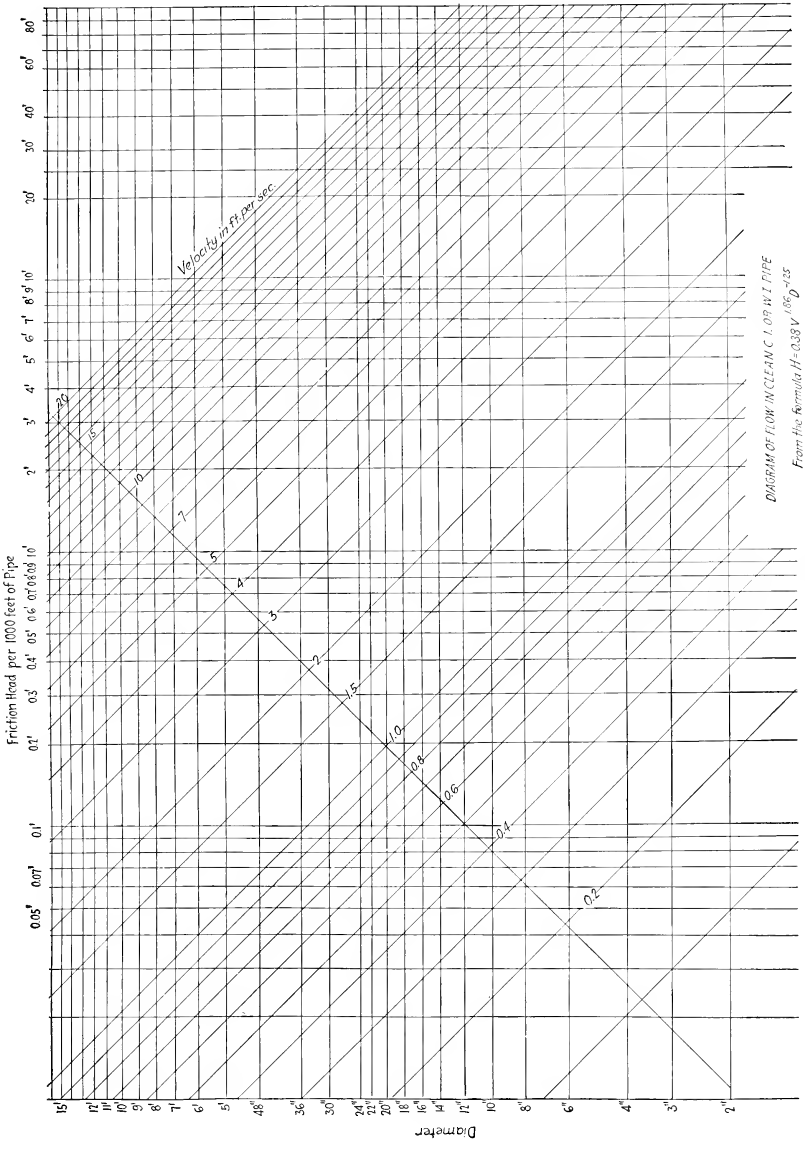

Friction Head per 1000 feet of Pipe
0.05' 0.07' 0.1' 0.2' 0.3' 0.4' 0.5' 0.6' 0.7' 0.8' 0.9' 1.0' 2' 3' 4' 5' 6' 7' 8' 9' 10' 20' 30' 40' 60' 80'
Velocity in ft. per sec.
20 15 10 7 5 4 3 2 1.5 1.0 0.8 0.6 0.4 0.2
Diameter
15' 12' 11' 10' 9' 8' 7' 6' 5' 48" 36" 30" 24" 22" 20" 18" 16" 14" 12" 10" 8" 6" 4" 3" 2"
DIAGRAM OF FLOW IN CLEAN C. I. OR W. I. PIPE
From the formula $H = 0.38\,V^{1.86}D^{-1.25}$

Fig. 29.

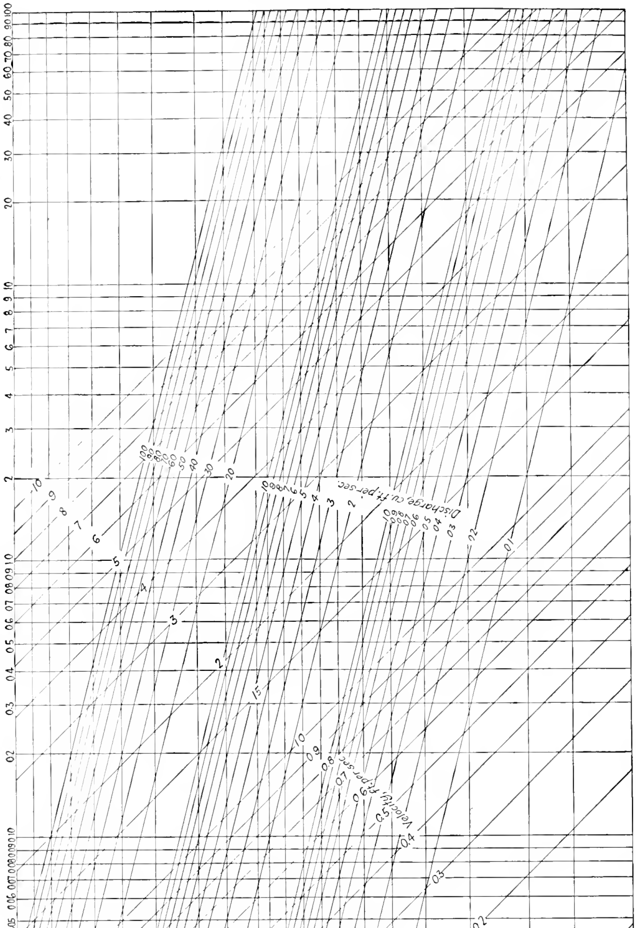


Diagram for the two equations $\left\{ \begin{array}{l} H = 0.38 V^{1.86} D^{-1.25} \\ Q = 0.7854 D^2 V \end{array} \right.$

Fig. 30.

If $x = \mu_1 \log H$ and $y = \mu_2 \log d$, there results a system of parallel lines for V

$$\frac{x}{\mu_1} + 1.25\frac{y}{\mu_2} - \log 0.38 - 1.86 \log V = 0$$

Figure 29 shows the completed diagram with $\mu_2 = 1.25$ and $\mu_1 = 1.0$. The normal bisects the angle between the axes and the scale on it is

$$p = \frac{1}{\sqrt{2}}(\log 0.38 + 1.86 \log V)$$

If the line system for V is to be drawn, it is of course not necessary to draw the normal since the lines of the system cross the X axis in points determined by the scale

$$p = \log 0.38 + 1.86 \log V$$

upon eliminating $\log V$ between the two logarithmic equations there results

$$1.86 \log Q = 1.86 \log \frac{0.7854}{0.38} + 4.97 \log d + \log H$$

and since $x = \log H$, $y = 1.25 \log d$, the system of parallel straight lines for Q is

$$x + \frac{4.97}{1.25}y = 1.86 \log Q - 1.86 \log \frac{0.7854}{0.38}$$

The necessary lines are added to Fig. 29 in Fig. 30. The angle for the system Q is 75° 53′.

Example 16.—The velocity V with which a jet of steam issues from a turbine nozzle having a friction factor Y is

$$V = 223.8\sqrt{(1 - Y)(H_1 - H_2)}$$

where $(H_1 - H_2)$ is the "Heat drop" or number of

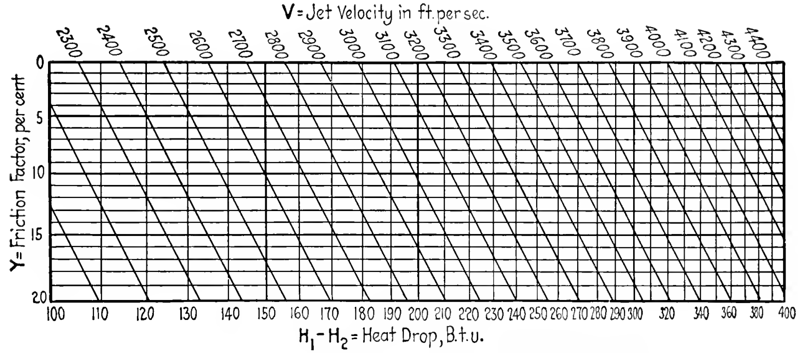


Diagram for Steam Jet Velocities, $V = 223.8\sqrt{(1 - Y)(H_1 - H_2)}$

FIG. 31.

The discharge Q is equal to the velocity of flow multiplied by the cross-section of the stream. For a circular pipe of diameter d the discharge is

$$Q = \frac{\pi}{4}d^2V$$

It is possible to supplement the diagram of Fig. 29 by new lines which will give the discharge. The example illustrates a general method available for use when four variables occur in this way in two equations. Since

$$\log Q = \log 0.7854 + 2 \log d + \log V$$

and

$$\log H + 1.25 \log d - \log 0.38 - 1.86 \log V = 0$$

British thermal units of energy available. Figure 31 shows a diagram for this formula with the following analysis:

$$\log (H_1 - H_2) + \log (1 - Y) + 2 \log 223.8 - 2 \log V = 0$$

If

$$x = \mu_1 \log (H_1 - H_2)$$
$$y = \mu_2 \log (1 - Y)$$

the parallel lines for V have as their equation

$$\mu_2 x + \mu_1 y - \mu_1\mu_2[2 \log V - 2 \log 223.8] = 0$$

The normal is located from

$$\cos \alpha = \frac{\mu_2}{\sqrt{\mu_1^2 + \mu_2^2}} \text{ and } \sin \alpha = \frac{\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}}$$

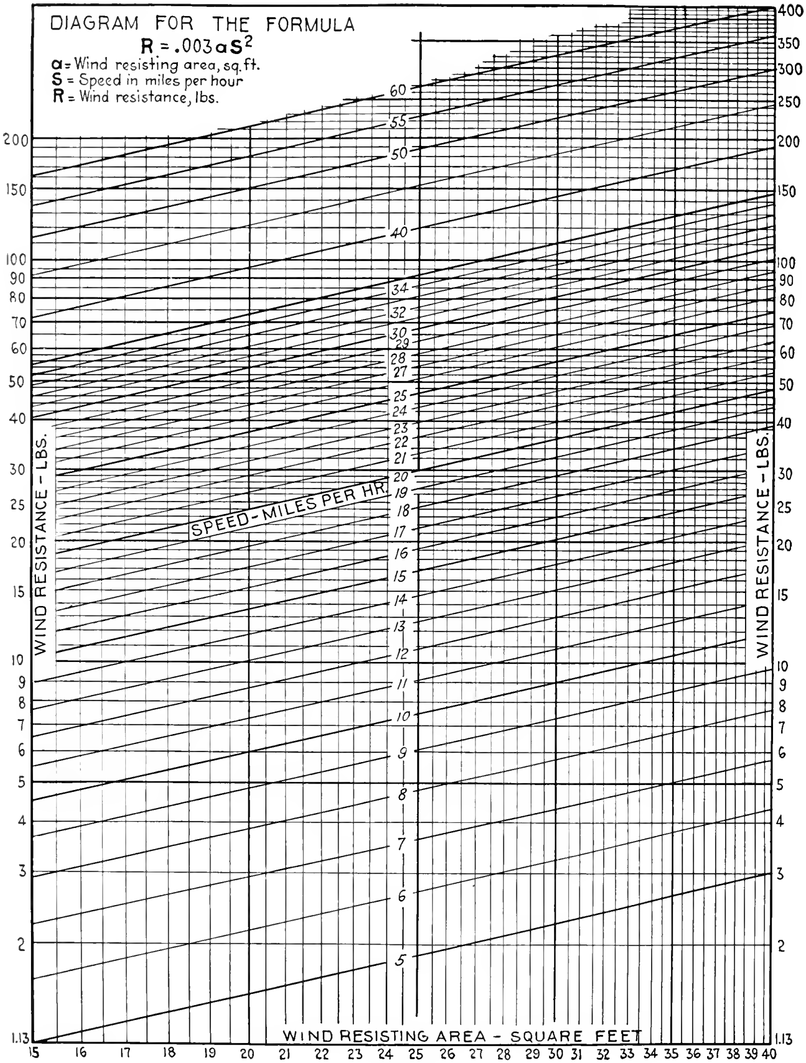


DIAGRAM FOR THE FORMULA
R = .003aS²
a = Wind resisting area, sq. ft.
S = Speed in miles per hour
R = Wind resistance, lbs.
WIND RESISTANCE - LBS.
SPEED-MILES PER HR.
WIND RESISTING AREA - SQUARE FEET
WIND RESISTANCE - LBS.

Fig. 32.

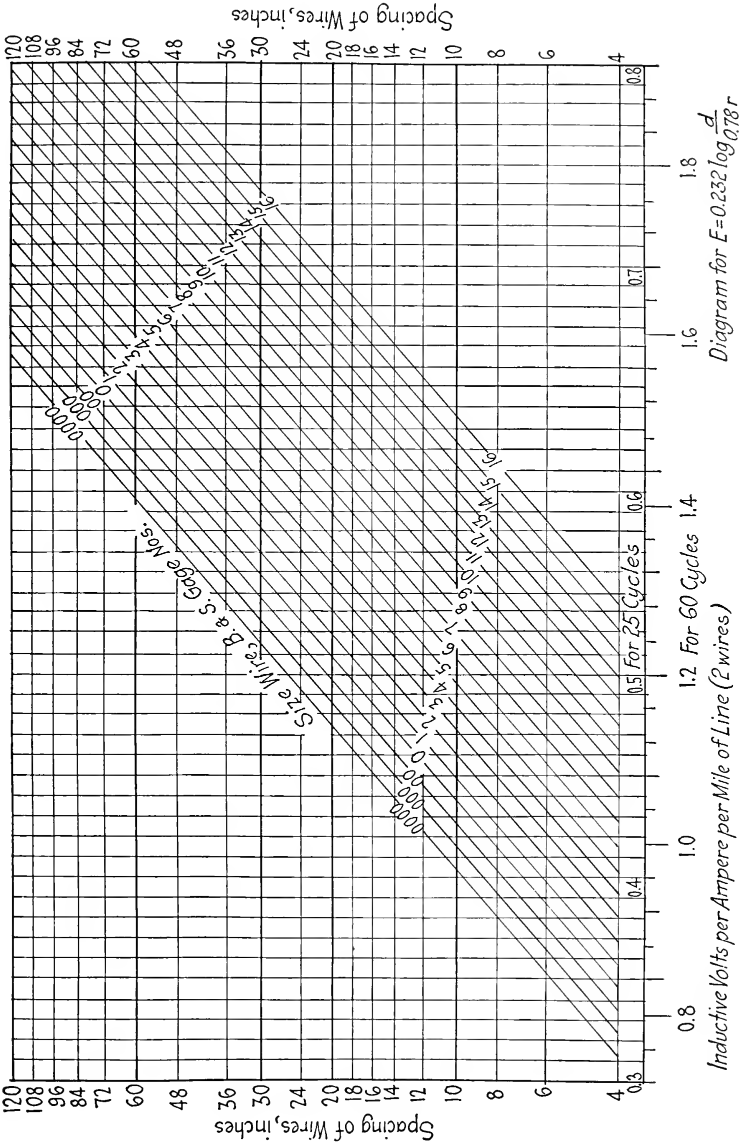

Spacing of Wires, inches
120 108 96 84 72 60 48 36 30 24 20 18 16 14 12 10 8 6 4
Size Wire, B. & S. Gage Nos.
0.3 0.4 0.5 For 25 Cycles 0.6 0.7 0.8
0.8 1.0 1.2 For 60 Cycles 1.4 1.6 1.8
Inductive Volts per Ampere per Mile of Line (2 wires)
Diagram for $E=0.232 \log \frac{d}{0.78r}$

FIG. 33.

and the scale on it is

$$p = \frac{\mu_1\mu_2}{\sqrt{\mu_1{}^2 + \mu_2{}^2}}[2 \log V - 2 \log 223.8]$$

The origin is not shown on the diagram.

Example 17.—An empirical formula giving the number of pounds of wind resistance R in an automobile offering a square feet of wind resisting area at S miles per hour is

$$R = 0.003aS^2$$

Passing to logarithms

$$\log a + 2 \log S + \log 0.003 - \log R = 0$$

If

$$x = \mu_1 \log a$$
$$y = \mu_2 \log R$$

then

$$\frac{x}{\mu_1} - \frac{y}{\mu_2} + 2 \log S + \log 0.003 = 0$$

the scale on the normal is

$$p = \frac{\mu_1\mu_2}{\sqrt{\mu_1{}^2 + \mu_2{}^2}}[2 \log S + \log 0.003]$$

The diagram is shown in Fig. 32.

Example 18.—In Fig. 33 is shown a diagram including parallel straight lines, for the Equation

$$E = 0.232 \log \frac{d}{0.78r}$$

which gives the inductive voltage E per ampere per mile of double wire for alternating currents, where r is the radius of the wire and d is the spacing, both in inches. As the size of the wire is usually expressed by the gauge the latter was used in constructing the diagram. To correct E for various frequencies the constant must be varied; the present diagram is drawn for both 25 and 60 cycles. It is not necessary to pass to logarithms in order to bring this equation into a form similar to type Equation (6)

$$\frac{E}{0.232} = \log d - \log 0.78r = 0$$

If

$$x = \mu_1\frac{E}{0.232}$$

and

$$y = \mu_2 \log d$$

the third system is $\frac{x}{\mu_1} - \frac{y}{\mu_2} + \log 0.78\,r = 0$

10. Hexagonal Diagrams.—For Equation (6) above, the resulting equation for the lines of the variable z_3 may be given a special form by setting $\mu_1 = \mu_2$ when the range of the values of z_1 and z_2 permits. The scale factor for the z_3 scale on the normal reduces then to $\frac{1}{\sqrt{2}}$. The factor $\frac{1}{\sqrt{2}}$ may be dispensed with by choosing the axes for the z_1 and z_2 scales at an angle of 120° and establishing the z_3 scale on the bisector of this angle. It can be proved from Fig. 34 that if from any point P perpendiculars are drawn to three scales there shown the following geometric relation holds

$$\overline{OM}_1 + \overline{OM}_2 = \overline{OM}_3$$

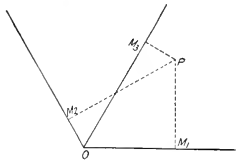


Fig. 34.

This relation is easily seen by observing that in Fig. 35, $\overline{AM}_3 = \overline{M_3B}$ so that $2\overline{OM}_3 = \overline{OA} + \overline{OB}$, but $\overline{OA} = 2\overline{OM}_2$ and $\overline{OB} = 2\overline{OM}_1$, whence the relation above

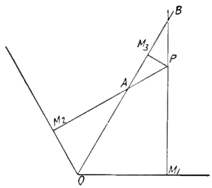


Fig. 35.

If now

$$\overline{OM}_1 = \mu f_1$$
$$\overline{OM}_2 = \mu f_2$$
$$\overline{OM}_3 = \mu f_3$$

it follows always that

$$f_1 + f_2 = f_3$$

for the values of z found at the corresponding points M_1, M_2, M_3.

This form of diagram is called the hexagonal form from the fact that the lines involved are the diagonals of a hexagon.

Example 19.—The formula of Example 15

$$H = 0.38\frac{V^{1.86}}{d^{1.25}}$$

may be readily represented by a hexagonal diagram if written

$$(\log H - \log 0.38) + 1.25 \log d = 1.86 \log V$$

Figure 36 shows the completed diagram. While the scale factors of all three scales must be the same, the coefficients 1.00, 1.25, and 1.86 determine the unit length of the scales. The constant log 0.38 in the H

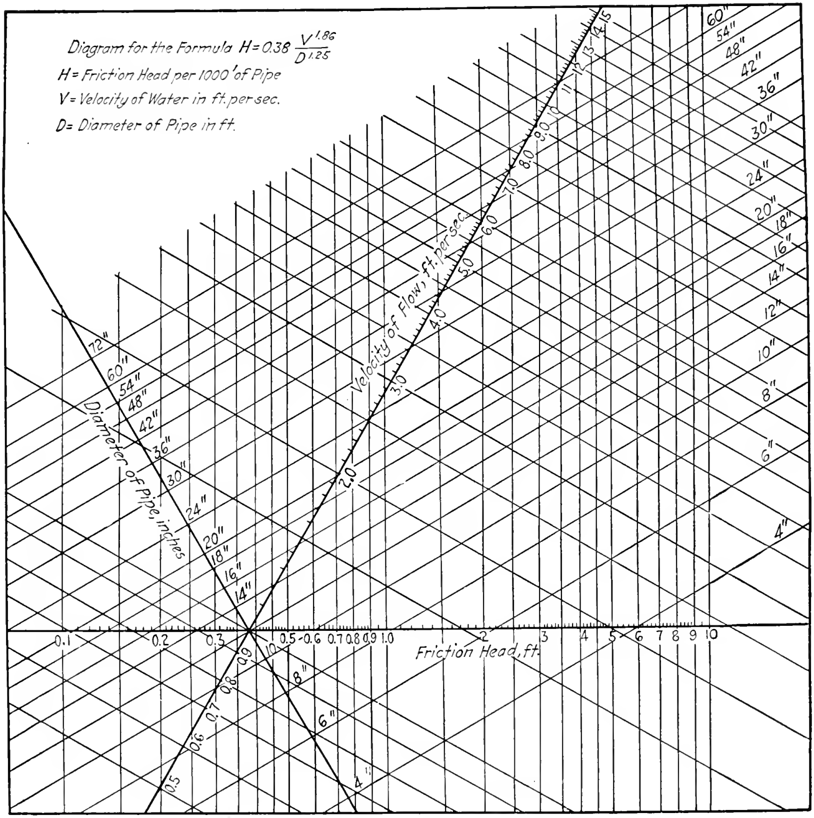


Diagram for the Formula $H=0.38\frac{V^{1.86}}{D^{1.25}}$
H = Friction Head per 1000' of Pipe
V = Velocity of Water in ft. per sec.
D = Diameter of Pipe in ft.
Velocity of Flow, ft. per sec.
Diameter of Pipe, inches
Friction Head, ft.
72" 60" 54" 48" 42" 36" 30" 24" 20" 18" 16" 14" 12" 10" 8" 6" 4"
0.1 0.2 0.3 0.5 0.6 0.7 0.8 0.9 1.0 2 3 4 5 6 7 8 9 10
0.5 0.6 0.7 0.8 0.9 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10 11 12 13 14 15

FIG. 36.

function shows that the logarithmic scale for H must be moved to the left until 0.38 is at the origin.

The hexagonal diagram may be supplied with a sufficient number of scales to solve equations of the form

$$f_1 + f_2 + f_3 + \ . \ . \ . \ + f_n = 0 \qquad (7)$$

Write the equivalent system

$$\begin{aligned} f_1 + f_2 &= h_0 \\ h_0 + f_3 &= h_1 \\ h_1 + f_4 &= h_2 \\ &\ldots\ldots \\ h_{n-3} + f_{n-1} &= -f_n \end{aligned}$$

On a suitably inscribed diagram enter with z_1 and z_2 and obtain a temporary point M on a blank scale. Then from the point where the z_3 perpendicular cuts

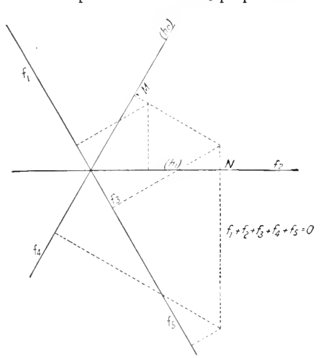


FIG. 37.

the perpendicular from M drop a perpendicular to locate N and proceed in this way until z_n is reached. The arrangement of scales for $n = 5$ is shown in Fig. 37. Another treatment of Equation (7) will be found in Article 21 of Chapter V.

Problem 1.—The illustrative examples of this chapter (5 to 18 inclusive) may in most cases be represented by diagrams of types other than those used. Investigate all feasible types for each formula given.

Problem 2.—The volume V of the frustrum of a cone of height h is

$$V = \frac{\pi}{12} h[D^2 + Dd + d^2]$$

where D and d are the diameters of the bases. Using D and d as z_1 and z_2 show how the system of curves for $z_3 = \frac{V}{h}$ may become a family of concentric circles and construct the diagram.

Problem 3.—Boussinesq's approximate formula for the perimeter of an ellipse L with semi-axes a and b is

$$L = \pi[3/2(a + b) - \sqrt{ab}]$$

Show that with a and b as z_1 and z_2 the curves for $z_3 = L$ may become circles tangent to both coordinate axes if a suitable angle is chosen for YOX.

Problem 4.—Draw all feasible diagrams for

$$\frac{T_1}{T_2} = e^{f\theta}$$

the ratio of belt or rope tensions T_1 and T_2 for a coefficient of friction f and an angle of wrap θ.

Problem 5.—Determine the corresponding formula when a set of observations of two variables result in a parabola symmetrical to the Y axis when plotted on logarithmic cross-section paper.

Problem 6.—Construct a diagram for the cubic equation $z^3 + pz + q = 0$ similar to that of Example 7, page 13, with regular scales for p and q on the axes.

Problem 7.—The capacity of a silo is given by K. J. T. Eckblaw as

$$C = \frac{d^2}{256}\left(\frac{h^2}{20} + 2.6h - 4\right)$$

where C is the capacity in tons, h the height in feet and d the diameter in feet.

(a) Construct a diagram using parallel straight line systems.

(b) Construct a diagram using a radial straight line system.

Problem 8.—F. W. Taylor gives the expression for the pressure upon a cutting tool when cutting cast iron

$$P = CD^{14/15} F^{3/4}$$

where P is in pounds, D is the depth of cut in inches and F is the feed in inches. The quantity C is taken as 45,000 for soft cast iron up to 69,000 for hard cast iron. Construct a convenient diagram.

Problem 9.—The expression $P_m = 3.463P_1(R^{.29} - 1)$ is used in determining the mean effective pressure P_m when air is compressed from an initial absolute pressure P_1 pounds per square inch and R is the ratio of the final to the initial pressure. Devise a diagram with parallel straight lines.

Problem 10.—In problems involving compound interest the expression $R = (1 + r)^n$ is the basis of all such computations. Devise a useful diagram for this expression.

Problem 11.—Devise and construct a convenient diagram which may be used to determine the correct revolutions per minute for pieces of work of various diameters (in inches) when certain cutting speeds (in feet per minute) are desired in various rotary machines.

Problem 12.—Look up the formula by Grashof for the flow of air through orifices and construct a diagram for use only within the limits for which the formula is applicable.

Problem 13.—Construct a diagram for the two formulas of Example 15 using ordinary logarithmic cross-section paper with equal scale factors on the axes. Plot H on OX and Q on OY.

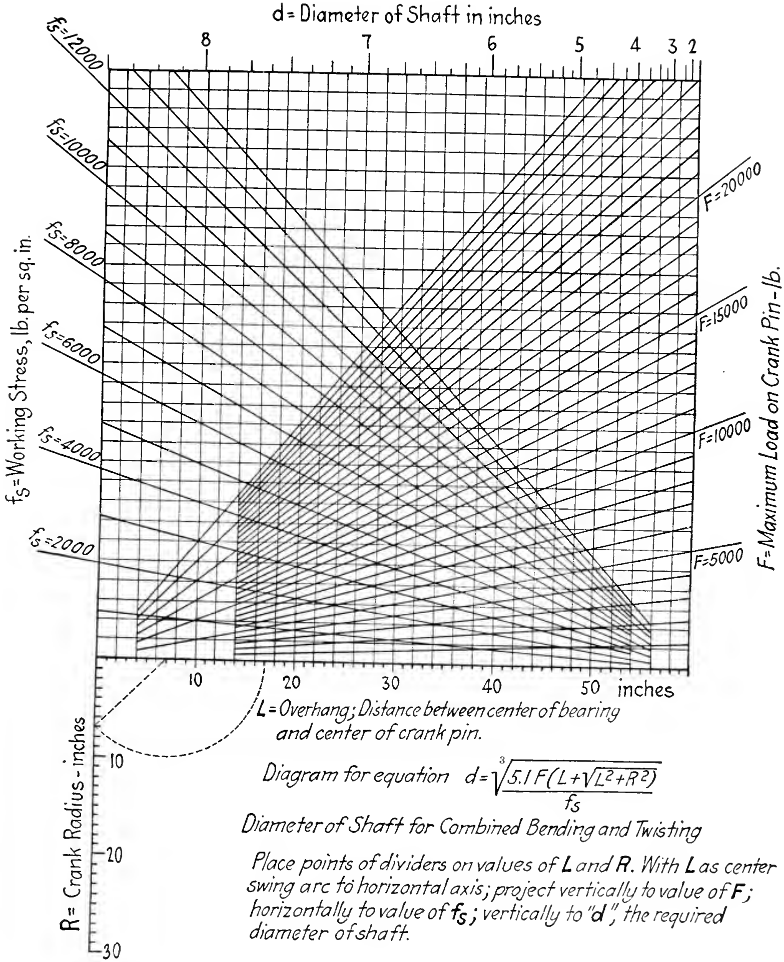

d = Diameter of Shaft in inches
8
7
6
5
4
3
2
fs=12000
fs=10000
fs=8000
fs=6000
fs=4000
fs=2000
fs = Working Stress, lb. per sq. in.
F=20000
F=15000
F=10000
F=5000
F = Maximum Load on Crank Pin-lb.
10
20
30
40
50 inches
L = Overhang; Distance between center of bearing and center of crank pin.
R = Crank Radius - inches
10
20
30
Diagram for equation $d = \sqrt[3]{\frac{5.1 F(L+\sqrt{L^2+R^2})}{f_s}}$
Diameter of Shaft for Combined Bending and Twisting
Place points of dividers on values of L and R. With L as center swing arc to horizontal axis; project vertically to value of F; horizontally to value of fs; vertically to "d", the required diameter of shaft.

FIG. 37*a*.

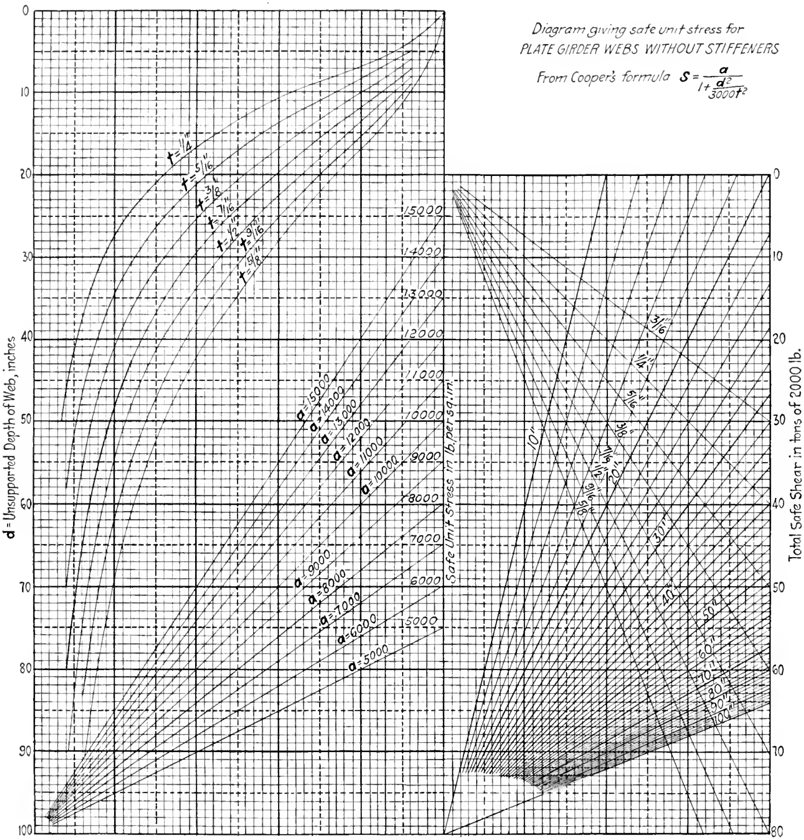


Diagram giving safe unit stress for
PLATE GIRDER WEBS WITHOUT STIFFENERS
From Cooper's formula $S = \frac{a}{1+\frac{d^2}{3000t^2}}$
d = Unsupported Depth of Web, inches
Safe Unit Stress in lb. per sq. in.
Total Safe Shear in tons of 2000 lb.
t = 1/4"
t = 5/16"
t = 3/8"
t = 7/16"
t = 1/2"
t = 9/16"
t = 5/8"
a = 15000
a = 14000
a = 13000
a = 12000
a = 11000
a = 10000
a = 9000
a = 8000
a = 7000
a = 6000
a = 5000
15000
14000
13000
12000
11000
10000
9000
8000
7000
6000
5000
3/16"
1/4"
5/16"
3/8"
7/16"
1/2"
9/16"
5/8"
10"
20"
30"
40"
50"
60"
70"
80"
90"
100"
0
10
20
30
40
50
60
70
80
90
100

Fig. 37b.

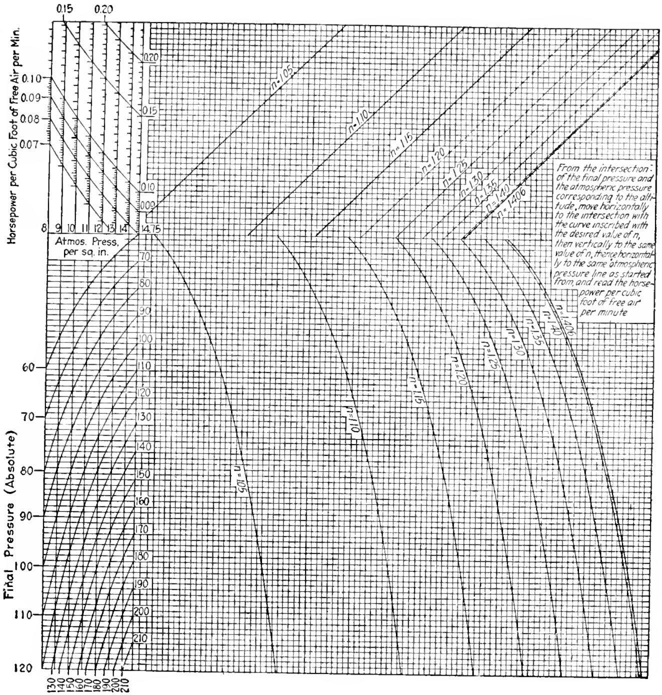

Horsepower per Cubic Foot of Free Air per Min.
0.15
0.20
0.10
0.09
0.08
0.07
0.20
0.15
0.10
0.09
8 9 10 11 12 13 14 14.75
Atmos. Press. per sq. in.
Final Pressure (Absolute)
60
70
80
90
100
110
120
70 80 90 100 110 120 130 140 150 160 170 180 190 200 210
130 140 150 160 170 180 190 200 210
n=1.05
n=1.10
n=1.15
n=1.20
n=1.25
n=1.30
n=1.35
n=1.40
n=1.406
From the intersection of the final pressure and the atmospheric pressure corresponding to the altitude, move horizontally to the intersection with the curve inscribed with the desired value of n, then vertically to the same value of n, thence horizontally to the same atmospheric pressure line as started from, and read the horsepower per cubic foot of free air per minute

Fig. 37c.

Problem 14.—Construct a diagram for the formula

$$x = 2\pi f\left(80 + 741.1 \log \frac{D}{r}\right)10^{-6}$$

which gives the inductive reactance x in ohms of a transmission line when D is the spacing of the wires in feet, r is the radius of the wire in inches and f the frequency. Take f as 60 cycles or some desired frequency and include a scale for wire sizes in connection with the r scale.

Problem 15.—Construct a diagram for the expression

$$H = 0.0274V^2 + 0.0141LV^{1.83}$$

for the friction loss H in condenser tubes L feet long when the water velocity is V feet per second through the tubes. L is in feet of water head.

Problem 16.—Analyze the methods of construction used in Figs. 37*a*, 37*b*, 37*c* and 37*d*.

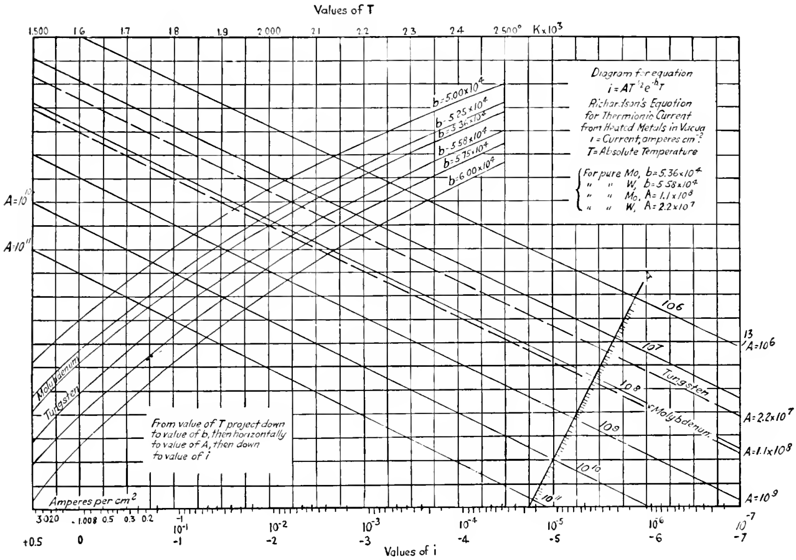


FIG. 37*d*.

CHAPTER III

ALIGNMENT DIAGRAMS OR COLLINEAR NOMOGRAMS[1]

11. General Type of Equation and Method of Treatment.—There will be considered in this chapter a great class of formulas which may be written in the determinant form

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \qquad (8)$$

in which f_1 and g_1 are functions of z_1, f_2 and g_2 functions of z_2, etc. Such knowledge of the elementary properties of determinants of the third order as may be gained from the reading of Appendix A will be assumed.

A distinguishing characteristic of the determinant of Equation (8) is the presence of the same variable in the elements of each row. There are many practical formulas which may be reduced to this form and their diagrammatic representation is of much value. Such formulas lead to a new form of diagram which will be called the *alignment diagram* because its key is the alignment of three points.

It is proved in analytic geometry that if three points P_1, P_2, P_3 with the coordinates (x_1y_1), (x_2y_2), (x_3y_3), respectively, lie on a straight line (are collinear) the coordinates satisfy the relation

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3 = 0$$

which expresses the fact that the point $P_3(x_3y_3)$ lies on the line joining the points $P_1(x_1y_1)$ and $P_2(x_2y_2)$ and whose equation is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The problem is then to establish a relation between the variables z_1, z_2, z_3 of Equation (8) and the position of three corresponding and inscribed variable points in the plane such that whenever three values of z are solutions of Equation (8) there shall correspond three such points in a straight line. When this relation is established, a straight edge applied through two points marked with known values of $z_i\,z_j$ must pass through one or more points marked with the value of z_k which satisfies Equation (8).

This problem is most easily solved by using the parametric form of the equations of plane curves where z is the parameter. The equations

$$x = f(z) \qquad\qquad y = g(z)$$

are the *parametric equations* of a plane curve C. For every value of the parameter z they determine a point P on that curve. Three such sets of equations will likewise determine three curves and the forms of the curves will depend on the nature of the functions f and g.

Three sets of such parametric functions may always be determined directly from Equation (8). If the three pairs of equations

$$\begin{aligned} x_1 &= f_1 & \qquad y_1 &= g_1 \\ x_2 &= f_2 & \qquad y_2 &= g_2 \\ x_3 &= f_3 & \qquad y_3 &= g_3 \end{aligned}$$

are formed, using the elements of the determinant of Equation (8) in the order shown, they may be considered as the parametric equations of three plane curves C_1, C_2, C_3. These equations will be called the *defining equations*. When the curves are plotted, points are inscribed with corresponding values of z and thus three curved function scales are obtained. There is then established a direct correspondence between values of z and points P on the plane curves C.

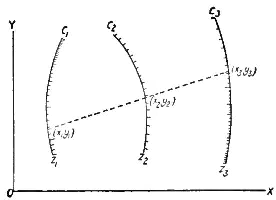


FIG. 38.

[1] Appendix A should be read before this chapter.

(See Fig. 38.) It is seen therefore that if $x_i y_i (i = 1, 2, 3)$ in the equation

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

are the coordinates of the points of the curves defined by the three pairs of equations above, then Equation (8) is always satisfied by values of z_1, z_2, z_3, which determine collinear points.

When an engineering formula or equation in three variables is given for which a diagram is desired, the first step is to write it in the determinant form. Equation (8) is the general type equation in three variables for which corresponds a collinear nomogram. Usually, however, an equation or formula does not present itself in a determinant form nor especially in this rather simple determinant form. Since it is always necessary to establish the defining equations before constructing a diagram it is very desirable to become familiar with the necessary determinant notation at once. Equation (8) with all the elements of the last column unity is called the *reduced determinant form*. It is almost always necessary to establish a *first determinant form* for any given equation and then transform it by the laws of determinants into the desired form above.

There is no general method by which any equation of the form

$$f(z_1 z_2 z_3) = 0$$

may be given a first determinant form and in fact not all equations in three variables may be written in that determinant form.

Special cases of Equation (8) have been studied and the necessary and sufficient conditions developed for identifying a given equation with them. The work involves partial differentiation and is not usually needed in practice.[1]

12. Diagrams with Three Parallel Straight Scales. In the expanded form of Equation (8) which is

$$f_1 g_2 + f_2 g_3 + f_3 g_1 - f_2 g_1 - f_3 g_2 - f_1 g_3 = 0 \quad (9)$$

should one or more of the functions f or g reduce to a constant and especially to zero the equation becomes much simplified. For example, the equation

$$f_1 + f_2 + f_3 = 0 \quad (6)$$

previously discussed in Chapter II, Article 9, results if

$$g_1 = -1, \; g_2 = 1, \; g_3 = 0, \text{ and } f_3 \equiv -\frac{f_3}{2}$$

[1] Clark, J., Théorie Générale des Abaques d'Alignment de toute Ordre, *Révue de Mécanique*, 1907, No. 39. Also d'Ocagne, Nos. 152–153, Traité de Nomographie.

A corresponding first determinant form of Equation (6) is

$$\begin{vmatrix} f_1 & -1 & 1 \\ f_2 & 1 & 1 \\ -\frac{f_3}{2} & 0 & 1 \end{vmatrix} = 0$$

Although this is a reduced form of the equation, in the sense defined above, it is usual to write this equation in the form resulting from an interchange of the first two columns thus

$$\begin{vmatrix} -1 & f_1 & 1 \\ 1 & f_2 & 1 \\ 0 & -\frac{f_3}{2} & 1 \end{vmatrix} = 0 \quad (10)$$

The defining equations[2] of the three corresponding scales are

$$\begin{aligned} x &= -1 & \quad y &= f_1 \\ x &= 1 & \quad y &= f_2 \\ x &= 0 & \quad y &= -\frac{f_3}{2} \end{aligned}$$

and the scales are consequently graduated on three equidistant parallel lines. This is perhaps the simplest form of collinear nomogram or diagram of alignment.

Example 20.—By the method of "end areas" the volume of earthwork per station on railway and highway construction is given by the formula

$$KV = (p_1 + p_2)$$

where V = volume in cubic yards,
K = a constant depending on the length of section and scale,
p_1 and p_2 are average planimeter readings in square inches from the cross-section drawings.

Comparing this formula with Equations (6) and (10) it is seen that the necessary defining equations are

$$\begin{aligned} x &= -1 & \quad y &= p_1 \\ x &= 1 & \quad y &= p_2 \\ x &= 0 & \quad y &= \frac{KV}{2} \end{aligned}$$

The diagram (for the scale of cross-sections 4 feet = 1 inch) may be constructed with the vertical unit one-tenth of an inch and the horizontal unit 5 inches. If desired the scales may be broken and repeated to avoid unduly enlarging the diagram. See Fig. 39.

It usually happens that for the range of values of the variables involved in the Equation (6) it is neces-

[2] Henceforth it will be sufficiently clear that three curves are under consideration without using subscripts to distinguish the coordinates of their respective points.

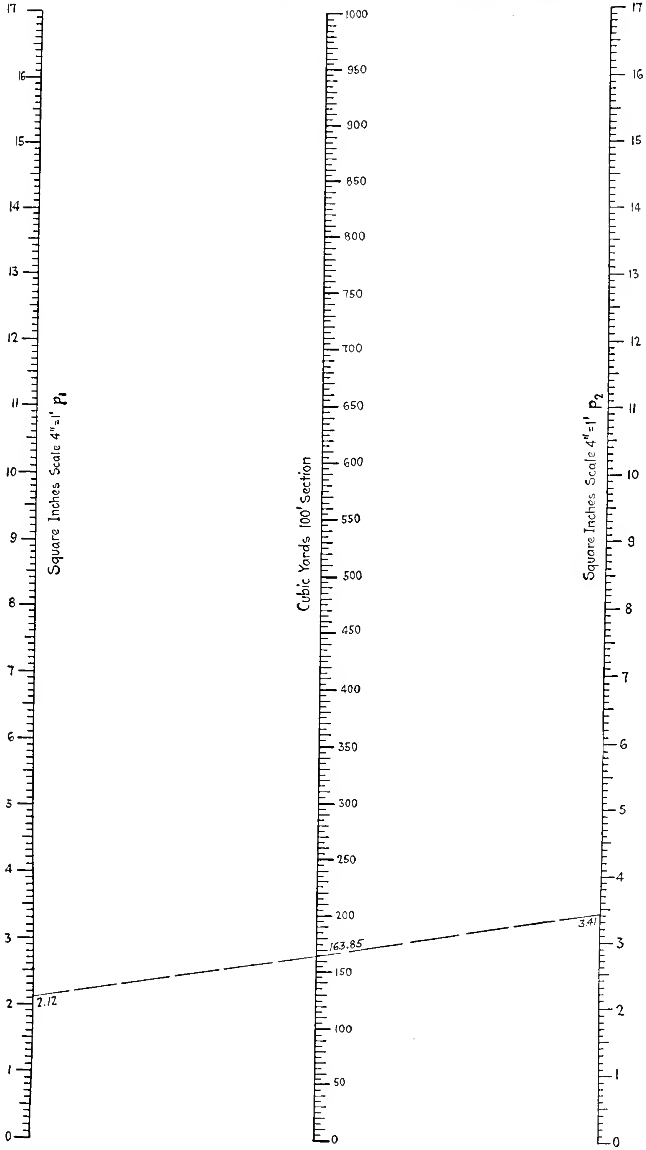


FIG. 39.—Diagram for $KV = (p_1 + p_2)$ V = Volume of Earthwork, cu. yds. p_1 and p_2 = Average Planimeter Readings, sq. in.

sary to introduce scale factors and sometimes it is desirable to establish the scales at unequal distances. Suppose that it is desired to introduce the scale factors μ_1 and μ_2 on the parallel scales for z_1 and z_2 and to establish these scales at distances δ_1 and δ_2 from the Y axis. It is then necessary to determine how the third scale shall be graduated.

The new defining equations for the first two scales would necessarily be written

$$x = -\delta_1 \qquad y = \mu_1 f_1$$
$$x = \delta_2 \qquad y = \mu_2 f_2$$

and it may be assumed temporarily that the third scale equations will have the form

$$x = F_3 \qquad y = G_3$$

where F_3 and G_3 are to be functions of f_3 and involve the new constants.

To determine F_3 and G_3 so that points originally corresponding to any set of solutions of Equation (6) shall remain collinear in the changed diagram it is necessary that the equation

$$\begin{vmatrix} -\delta & \mu_1 f_1 & 1 \\ \delta_2 & \mu_2 f_2 & 1 \\ F_3 & G_3 & 1 \end{vmatrix} = 0$$

shall be satisfied by virtue of Equation (6). Upon expanding this equation and substituting the value of f_1 from equation (6) there results

$$(\mu_1\delta_2 - \mu_1 F_3)(f_2 + f_3) - (F_3 + \delta_1)\mu_2 f_2 + (\delta_1 + \delta_2)G_3 = 0$$

Since this equation must hold for any values of the independent variables z_2 and z_3, then the coefficient of f_2 and the term not involving f_2 must vanish identically, that is

$$\mu_1\delta_2 - \mu_1 F_3 - \mu_2\delta_1 - \mu_2 F_3 = 0$$

and $$G_3(\delta_1 + \delta_2) + (\mu_1\delta_2 - \mu_1 F_3)f_3 = 0$$

whence $$F_3 = -\frac{\delta_1\mu_2 - \delta_2\mu_1}{\mu_1 + \mu_2} \qquad G_3 = -\frac{\mu_1\mu_2 f_3}{\mu_1 + \mu_2}$$

and the defining equations of the third scale are

$$x = \frac{\delta_2\mu_1 - \delta_1\mu_2}{\mu_1 + \mu_2} \qquad y = -\frac{\mu_1\mu_2 f_3}{\mu_1 + \mu_2}$$

It is seen that when $(\delta_2\mu_1 - \delta_1\mu_2) = 0$, the new scale will remain on the Y axis and the constants may usually be so chosen that this is true. It is to be observed also that the scale factor of the third scale is independent of δ_1 and δ_2. Frequently δ_1 and δ_2 may be chosen equal in which case μ_1 and μ_2 must also be equal if the third scale is to remain on the Y axis; that is to say if the three scales are to be at equal distances. The quantity $\dfrac{\mu_1\mu_2}{\mu_1 + \mu_2}$ may be called the scale factor μ_3 of the third parallel scale.

As a check on the work the values of F_3 and G_3 above determined may be substituted in the last determinant equation with the result

$$\begin{vmatrix} -\delta_1 & \mu_1 f_1 & 1 \\ \delta_2 & \mu_2 f_2 & 1 \\ \dfrac{\delta_2\mu_1 - \delta_1\mu_2}{\mu_1 + \mu_2} & \dfrac{-\mu_1\mu_2 f_3}{\mu_1 + \mu_2} & 1 \end{vmatrix} = -\frac{\mu_1\mu_2(\delta_1 + \delta_2)}{\mu_1 + \mu_2}[f_1 + f_2 + f_3] = 0$$

It is well to point out here that the effect of the introduction of the above scale factors and the change of moduli is to apply a *projective transformation*[1] (see Appendix B) to the original geometric configuration. A projective transformation when applied to all the variable elements of the first two columns of such a third order (reduced) determinant has the effect of manipulating the elements of the determinant by the laws of determinants and the net result is always merely to multiply it by a constant. In the present case the constant is

$$-\frac{\mu_1\mu_2(\delta_1 + \delta_2)}{\mu_1 + \mu_2}$$

To understand how the above theory of the scale factors is applied, the formula for volume by "end areas" of Example 20 may be resumed. The use of a horizontal unit of 5 inches and a vertical unit of one-tenth of an inch was equivalent to the introduction of the values

$$\delta_1 = \delta_2 = 5,\ \mu_1 = \mu_2 = 1/10$$

in order to change the defining equations for the diagram to

$$x = -5 \qquad y = \frac{P_1}{10}$$
$$x = 5 \qquad y = \frac{P_2}{10}$$
$$x = 0 \qquad y = \frac{KV}{20}$$

It is to be observed strictly that in all the equations above f_3 is the value appearing in Equation (6).

By using a logarithmic transformation any equation of the form

$$z_1^{\alpha} = Kz_2^{\beta}z_3^{\gamma} \tag{11}$$

(α, β and γ = constants) may be written in the form of equation (6) thus

$$\alpha \log z_1 - \beta \log z_2 - \gamma \log z_3 - \log K = 0$$

The corresponding diagram has three parallel logarithmic scales defined by the equations

$$x = -1 \qquad y = \alpha \log z_1$$
$$x = 1 \qquad y = \beta \log z_2$$
$$x = 0 \qquad y = -1/2(\gamma \log z_3 + \log K)$$

[1] The projective transformation has the equations

$$x_1 = \frac{(\mu_1\delta_2 + \mu_2\delta_1)x + (\mu_1\delta_2 - \mu_2\delta_1)}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$

$$y_1 = \frac{2\mu_1\mu_2 y}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$

The following equation

$$z_1^{\alpha} z_2^{\beta} z_3^{\gamma} = \text{constant} \qquad (12)$$

may be similarly treated. The logarithm of the constant can of course be associated with any one of the variables desired for convenience in constructing and using the diagram.

Example 21.—An illustration of Equation (11) is afforded by the formula for the volume of a torus or ring of circular cross-section

$$V = 2.4674Dd^2$$

Taking logarithms of both sides of this equation it may be written

$$2 \log d + \log D - \log V + \log 2.4674 = 0$$

A corresponding reduced determinant form is therefore

$$\begin{vmatrix} -1 & 2 \log d & 1 \\ 1 & \log D & 1 \\ 0 & -\dfrac{\log 2.4674 - \log V}{2} & 1 \end{vmatrix} = 0$$

so that the three scales, when no scale factors are used, are defined as follows:

$$x = -1 \qquad y = 2 \log d$$
$$x = 1 \qquad y = \log D$$
$$x = 0 \qquad y = -\frac{\log 2.4674 - \log V}{2}$$

If the same limiting values are chosen for d and D it is seen that the scale for d will be twice as long as that for D. In order to have these scales of the same length and covering the same range of values and so arranged that both may be read with equal accuracy, choose

$$\mu_1 = 1 \qquad \mu_2 = 2$$

For convenience let $(\mu_1\delta_2 - \mu_2\delta_1) = 0$ so that $\delta_2 = 2\delta_1$ and the scale factor for the V scale will be

$$\mu_3 = \frac{\mu_1\mu_2}{\mu_1 + \mu_2} = \frac{2}{3}$$

The constant term log 2.4674 in the V function simply determines the initial point of the logarithmic scale for V, (see Fig. 40), for which the equations are

$$x = 0, \qquad y = \tfrac{1}{3}[\log V - \log 2.4674]$$

It frequently happens that two parallel scales will extend in opposite directions from the X axis and whenever this is so a displacement of the scales along their supports is desirable in order to dispose them to better advantage on the sheet. In the following example the K and R scales are started from a line making an angle of 45° with the X axis at the initial point of the A scale while the original distance between the scales is preserved. Geometrically this is the effect of carrying out upon the original diagram a projective transformation whose equations are

$$x_1 = x \qquad y_1 = x + y + 1$$

and consequently alignment is preserved.

Example 22.—The area of a segment of a circle of radius R and height H is given by the exact formula

$$A = R^2\left[\text{arc vers}\,\frac{H}{R} - \frac{\sqrt{2RH - H^2}}{R^2}(R - H)\right]$$

Since H appears always divided by R, write $\dfrac{H}{R} = K$, then

$$A = R^2[\text{arc vers}\,K - \sqrt{2K - K^2}(1 - K)]$$

and passing to logarithms

$$\log A = 2 \log R + \log [\text{arc vers}\,K - \sqrt{2K - K^2}(1 - K)]$$

so that the reduced determinant form is

$$\begin{vmatrix} -1 & \log A & 1 \\ 1 & -2 \log R & 1 \\ 0 & \tfrac{1}{2} \log[\text{arc vers}\,K - \sqrt{2K - K^2}(1 - K)] & 1 \end{vmatrix} = 0$$

Figure 41 shows the diagram for this formula constructed with unit scale factors. When $K = 1$ and when $K = 2$ the corresponding areas are respectively semi-circles and circles.

In most practical examples the displacement of a scale whose graduations increase in a downward direction from the X axis is best effected as in this example by simply starting it from a point above the X axis on a 45° line through the origin.

Example 23.—Figure 42 shows a diagram for the formula

$$P = C \cdot F^{3/4} \cdot D^{14/15}$$

which is given by F. W. Taylor[1] for the pressure on a tool when cutting cast iron, where

F = feed in inches,
P = pressure in pounds,
C = 45,000 for soft cast iron,
C = 69,000 for hard cast iron,
D = depth of cut in inches.

Passing to logarithms, the formula becomes

$$\tfrac{3}{4} \log F + \tfrac{14}{15} \log D = \log P - \log C$$

and the scales are defined by the equations

$$x = 1 \qquad y = \tfrac{3}{4} \log F$$
$$x = -1 \qquad y = \tfrac{14}{15} \log D$$
$$x = 0 \qquad y = \tfrac{1}{2}[\log P - \log C]$$

The constant C is associated with the P scale in order that its extreme values given above may be used in placing the graduations on the P scale. The diagram thus gives the maximum and minimum values of P for any D and F.

Example 24.—In correcting a barometer reading at a temperature t_1 to a temperature t for which the barometer is calibrated the correct reading in English units would be

$$h = h_1[1 - 0.000101(t_1 - t)]$$

[1] *Trans. A. S. M. E.*, vol. 28.

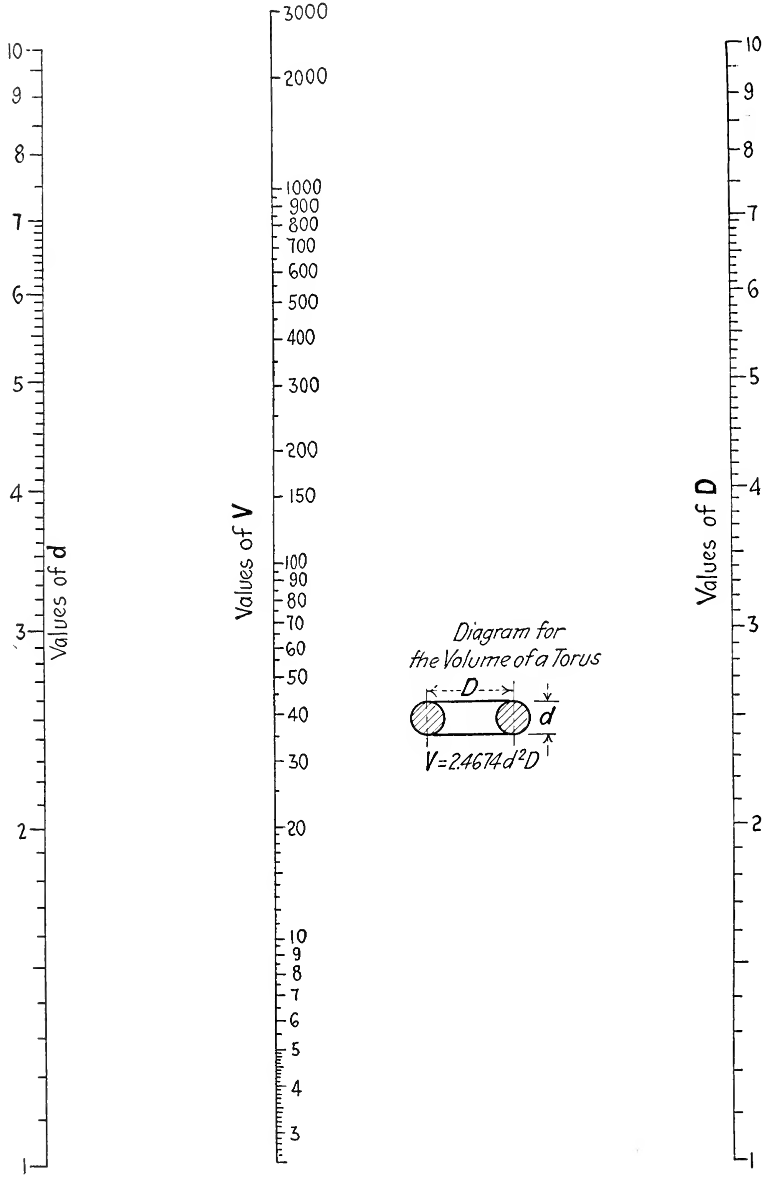

Values of d
Values of V
Values of D
Diagram for the Volume of a Torus
D
d
$V = 2.4674 d^2 D$
10
9
8
7
6
5
4
3
2
1
3000
2000
1000
900
800
700
600
500
400
300
200
150
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3

FIG. 40.

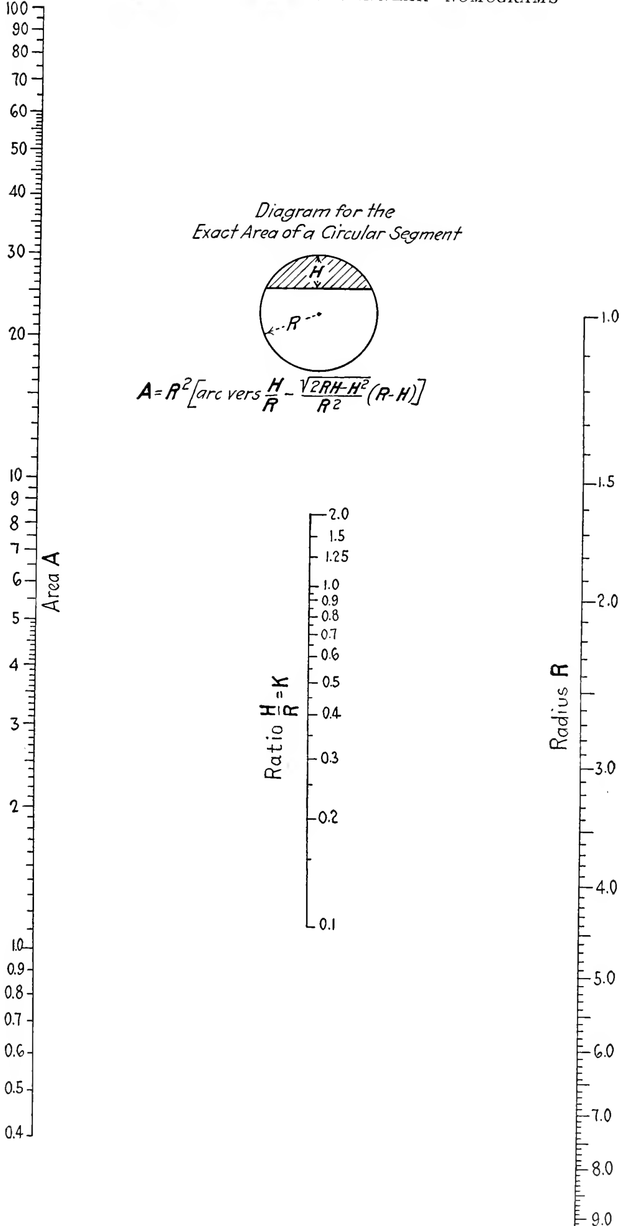


Diagram for the
Exact Area of a Circular Segment
H
R
$A = R^2\left[\text{arc vers}\,\frac{H}{R} - \frac{\sqrt{2RH-H^2}}{R^2}(R-H)\right]$
Area A
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
Ratio $\frac{H}{R}$=K
2.0
1.5
1.25
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Radius R
1.0
1.5
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0

Fig. 41.

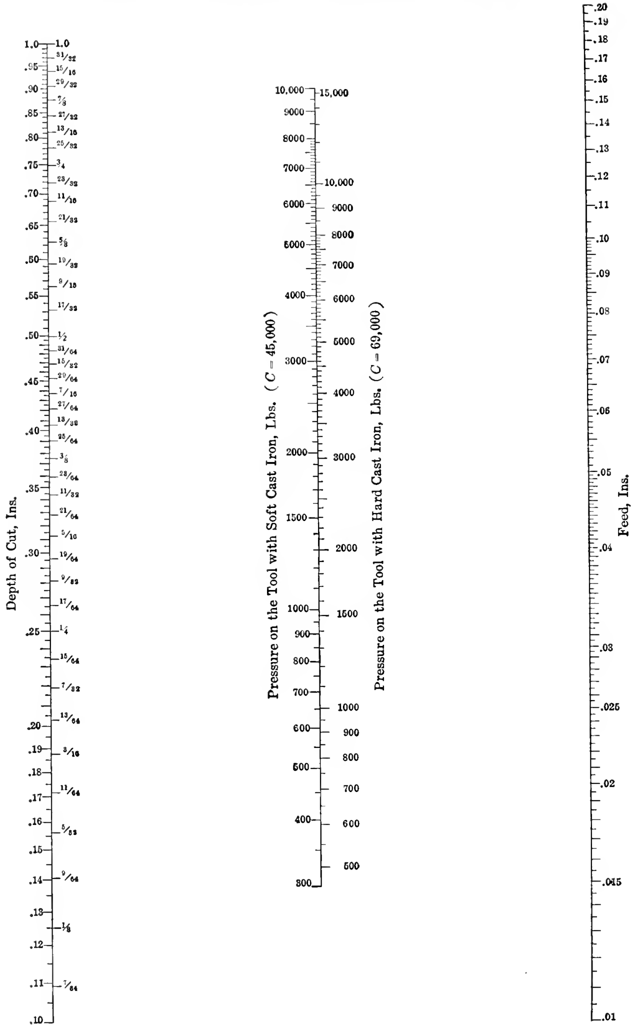


Fig. 42.—Diagram for $P = C F^{\frac{3}{4}} D^{\frac{14}{15}}$.

where h_1 is the observed height. The correction therefore is

$$E = h_1[\,0.000101\Delta t]$$

or, passing to logarithms,

$$\log E - \log 0.000101 = \log h_1 + \log \Delta t$$

If

$$x = -\delta_1 \qquad y = \mu_1 \log h_1$$
$$x = \delta_2 \qquad y = \mu_2 \log \Delta t$$

it would be desirable to apply the methods of Article 4 to determine the scale factors μ_1 and μ_2 in order to extend the scale resulting from the short range of numbers for h_1. If the h_1 and Δt scales are each to be 10 inches long and the following limits chosen:

h_1 from 26 to 31 inches of Mercury

Δt from 1 to 70 degrees Fahrenheit,

then

$$\mu_1 = 130 \quad \text{and} \quad \mu_2 = 5.42$$

Taking $(\mu_1\delta_2 - \mu_2\delta_1) = 0$ as before, there results

$$\frac{\delta_1}{\delta_2} = \frac{\mu_1}{\mu_2} = \frac{130}{5.42} = \frac{24}{1}$$

and for μ_3

$$\mu_3 = \frac{\mu_1\mu_2}{\mu_1 + \mu_2} = 5.19$$

The defining equations then are

$$x = -24 \qquad y = 130 \log h_1$$
$$x = 1 \qquad y = 5.42 \log \Delta t$$
$$x = 0 \qquad y = 5.19\,[\log E - \log 0.000101]$$

and the diagram appears in Fig. 43.

Example 25.—A modification by Grashof of Napier's Rule for the flow of steam through an orifice is sometimes used for steam nozzles in the following form

$$F = \frac{P^{0.97}A_0}{60}$$

where

F = flow of steam in pounds per second,

P = absolute initial pressure in pounds per square inch,

A_0 = area at throat in square inches.

If written in the logarithmic form, then

$$\log F + \log 60 = 0.97 \log P + \log A_0$$

In order to use the same units for the P and the A_0 scales let

$$\mu_2 = \frac{1}{0.97} \qquad \mu_1 = 1$$

then if

$$\frac{\delta_2}{\delta_1} = \frac{\mu_2}{\mu_1} = \frac{1}{0.97}, \quad \mu_3 = \frac{1}{1.97} = 0.508$$

the three scales are

$$x = 0.97 \qquad y = \log A_0$$
$$x = -1 \qquad y = \log P$$
$$x = 0 \qquad y = 0.508[\log F + \log 60]$$

See Fig. 44.

Example 26.—The Royal Automobile Club (England) automobile engine rating gives the rated horsepower, HP, of N cylinders of bore D inches and stroke S inches as

$$\text{HP} = \frac{D^2NS}{12}$$

or

$$\log\left[\frac{\text{HP}}{N}\right] + \log 12 = 2 \log D + \log S$$

In order to read the D and S scales on a diagram with equal ease, let

$$x = -\delta_1 \qquad y = \mu_1\, 2 \log D$$
$$x = \delta_2 \qquad y = \mu_2 \log S$$
$$x = 0 \qquad y = \mu_3\left[\log \frac{\text{HP}}{N} + \log 12\right]$$

if $\mu_1 = \frac{1}{2}$, $\mu_2 = 1$ then $\mu_3 = \frac{1}{3}$. See Fig. 45.

It is to be observed that when it is desirable to displace one or more of the parallel scales in a diagram it is not necessary to start the downward scales from a line making 45° with the X axis but any angle α whatever may be used. The equivalent projective transformation in the case of a diagram with scales originally at distances δ_1, δ_2 from the Y axis would have the equations

$$x_1 = x \qquad y_1 = (x + \delta_1)\tan\alpha + y$$

Equations of four variables of the form

$$f_1 + f_2 + f_3 + f_4 = 0 \tag{13}$$

may be represented by parallel scale diagrams and will be discussed in Chapter IV together with the more general type

$$f_1 + f_2 + f_3 + f_4 + \ \ldots\ + f_n = 0$$

13. Diagrams with Straight Scales and Two Only Parallel.—It is easily seen that the equations

$$x = 0 \qquad y = g_1$$
$$x = 1 \qquad y = g_2$$
$$x = f_3 \qquad y = 0$$

where g_1, g_2, f_3 are functions of z_1, z_2, z_3 respectively, would define a diagram in which there would be two parallel scales and a third straight scale perpendicular to them. What is the corresponding equation in three variables for which such a diagram would be useful?

Before deciding this question it is well to state that all equations or formulas are subject to a great variety of algebraic and other transformations: clearing of fractions, factoring, removal of radicals, separation or combination of constants, etc., which all tend to change the appearance of any given equation.

An equation corresponding to the particular type of collinear nomogram or diagram of alignment to be discussed is not difficult to establish for it is only

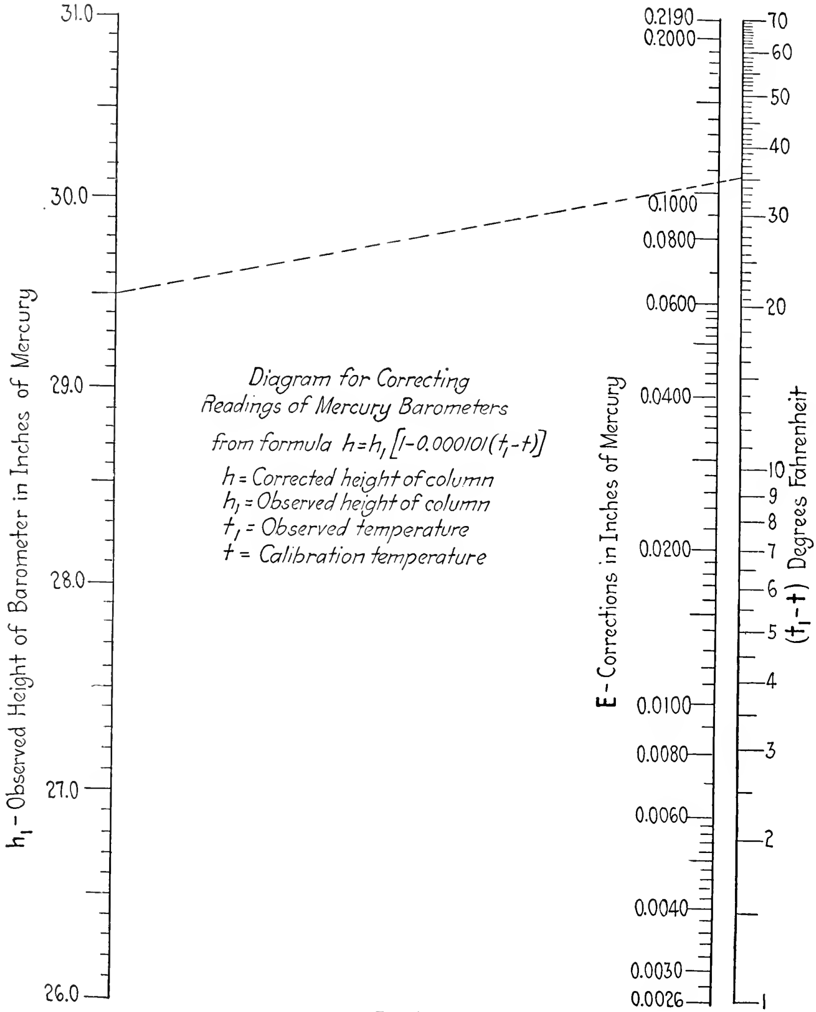


Diagram for Correcting
Readings of Mercury Barometers
from formula $h = h_1[1 - 0.000101(t_1 - t)]$
h = Corrected height of column
h_1 = Observed height of column
t_1 = Observed temperature
t = Calibration temperature
h_1 - Observed Height of Barometer in Inches of Mercury
31.0
30.0
29.0
28.0
27.0
26.0
E - Corrections in Inches of Mercury
0.2190
0.2000
0.1000
0.0800
0.0600
0.0400
0.0200
0.0100
0.0080
0.0060
0.0040
0.0030
0.0026
$(t_1 - t)$ Degrees Fahrenheit
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1

Fig. 43.

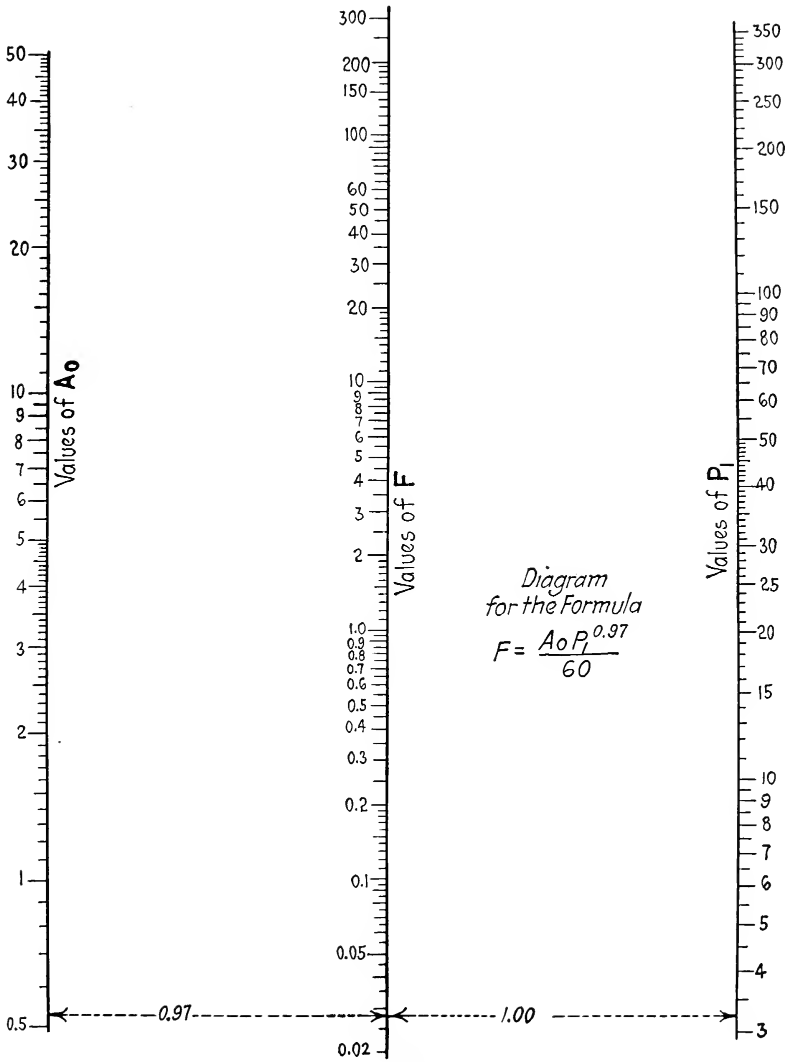

Values of A_0
50
40
30
20
10
9
8
7
6
5
4
3
2
1
0.5
Values of F
300
200
150
100
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.05
0.02
Values of P_1
350
300
250
200
150
100
90
80
70
60
50
40
30
25
20
15
10
9
8
7
6
5
4
3
Diagram for the Formula
$F = \frac{A_0 P_1^{0.97}}{60}$
0.97
1.00

FIG. 44.

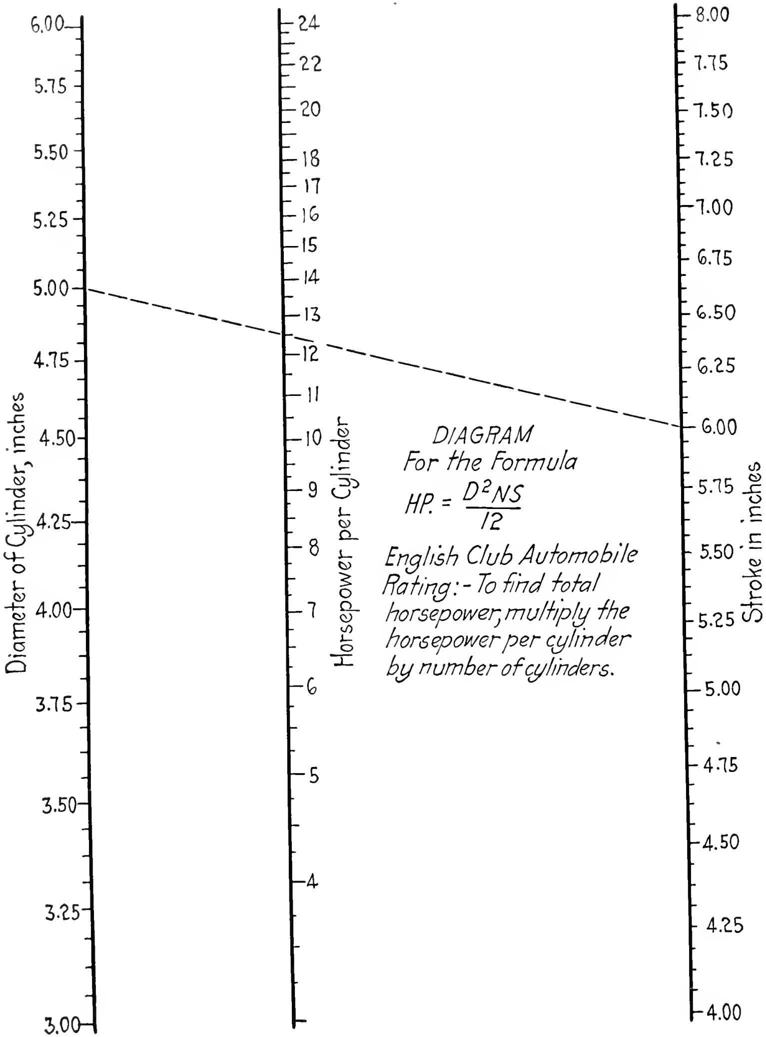

6.00
5.75
5.50
5.25
5.00
4.75
4.50
4.25
4.00
3.75
3.50
3.25
3.00
Diameter of Cylinder, inches
24
22
20
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
Horsepower per Cylinder
8.00
7.75
7.50
7.25
7.00
6.75
6.50
6.25
6.00
5.75
5.50
5.25
5.00
4.75
4.50
4.25
4.00
Stroke in inches
DIAGRAM
For the Formula
$HP. = \frac{D^2NS}{12}$
English Club Automobile Rating:- To find total horsepower, multiply the horsepower per cylinder by number of cylinders.

Fig. 45.

necessary to write the reduced determinant equation corresponding to the above defining equations

$$\begin{vmatrix} 0 & g_1 & 1 \\ 1 & g_2 & 1 \\ f_3 & 0 & 1 \end{vmatrix} = 0 \qquad (14)$$

But this reduced determinant equation may have resulted from any one of an indefinite number of first determinant forms all of which are equivalent under the laws of writing determinants without changing their values and the corresponding diagrams would either be alike or equivalent to each other by a projective transformation.

The expanded form of the determinant equation above is

$$f_3(g_1 - g_2) - g_1 = 0 \qquad (15)$$

but a more simple form of the expanded equation which would yield exactly the same type of diagram would be

$$f_1' - f_2'h_3' = 0 \qquad (16)$$

and the two equations are equivalent in view of the following relations

$$f_1' = g_2,\ f_2' = -g_1,\ h_3' = \frac{1 - f_3}{f_3}$$

The corresponding simple first determinant form for Equation (16) is

$$\begin{vmatrix} f_1' & 1 & 0 \\ -f_2' & 0 & 1 \\ 0 & 1 & h_3' \end{vmatrix} = 0$$

and the reduced determinant equation is obtained by adding column two to column three to form a new third column, then dividing the elements of the third row by $(1 + h_3')$ and then interchanging columns one and two and rows one and two. The resulting determinant equation is

$$\begin{vmatrix} 0 & -f_2' & 1 \\ 1 & f_1' & 1 \\ \dfrac{1}{1 + h_3'} & 0 & 1 \end{vmatrix} = 0 \qquad (17)$$

and it is only necessary to inspect the relations written above to identify this equation with Equation (14) which will henceforth be considered the reduced determinant form for an equation whose diagram has two parallel scales and a third perpendicular scale. On the other hand Equation (16) will be the type form for the expanded equation and Equation (17) is very useful to determine the defining scale equations. No determinant form of any equation should ever be used without first expanding it to check the correctness of the determinant. It is to be noticed that by transposing and passing to logarithms Equation (16) becomes identical in form with Equation (6) of Chapter II.

It is usually necessary to introduce scale factors for the construction of the parallel scales of Equation (14) and to move one scale a distance δ from the other. Let the corresponding defining equations for the changed parallel scales be

$$x = 0 \qquad y = \mu_1 g_1$$
$$x = \delta \qquad y = \mu_2 g_2$$

To determine the third scale equations one may proceed as in Article 12. Assume that the defining equations will have the form

$$x = F_3 \qquad y = G_3$$

where F_3 and G_3 are functions of f_3 and involve μ_1, μ_2, and δ. Then the changed determinant equation from (14) will be

$$\begin{vmatrix} 0 & \mu_1 g_1 & 1 \\ \delta & \mu_2 g_2 & 1 \\ F_3 & G_3 & 1 \end{vmatrix} = \mu_1 g_1 F_3 + \delta G_3 - \mu_2 g_2 F_3 - \delta \mu_1 g_1 = 0$$

But from the expanded form of Equation (14)

$$g_1 = \frac{g_2 f_3}{f_3 - 1}$$

and upon substituting this value of g_1 it is necessary that the previous equation become an identity for every value of g_2. Hence equating the coefficient of g_2 and the term not involving g_2 to zero there follows

$$F_3 = \frac{\delta \mu_1 f_3}{(\mu_1 - \mu_2) f_3 + \mu_2} \qquad G_3 = 0$$

and the third pair of defining equations becomes

$$x = \frac{\delta \mu_1 f_3}{(\mu_1 - \mu_2)\, f_3 + \mu_2} \qquad y = 0$$

It is seen that the scale on the X axis may be if desired obtained from a scale of the function f_3 by the methods of Chapter I, Article 2(*d*).

Example 27.—The formula of Francis

$$Q = 3.33BH^{3/2}$$

may also be given the determinant form

$$\begin{vmatrix} 0 & -Q & 1 \\ 1 & 3.33B & 1 \\ \dfrac{H^{3/2}}{H^{3/2} + 1} & 0 & 1 \end{vmatrix} = 0$$

and the defining equations are

$$x = 0 \qquad y = -\mu_1 Q$$
$$x = \delta \qquad y = \mu_2 3.33B$$
$$x = \frac{\delta \mu_1 H^{3/2}}{(\mu_1 - \mu_2) H^{3/2} + \mu_2 (H^{3/2} + 1)} \qquad y = 0$$

and if $\delta = 1$ there results for the H scale simply

$$x = \frac{\mu_1 H^{3/2}}{\mu_1 H^{3/2} + \mu_2} \qquad y = 0$$

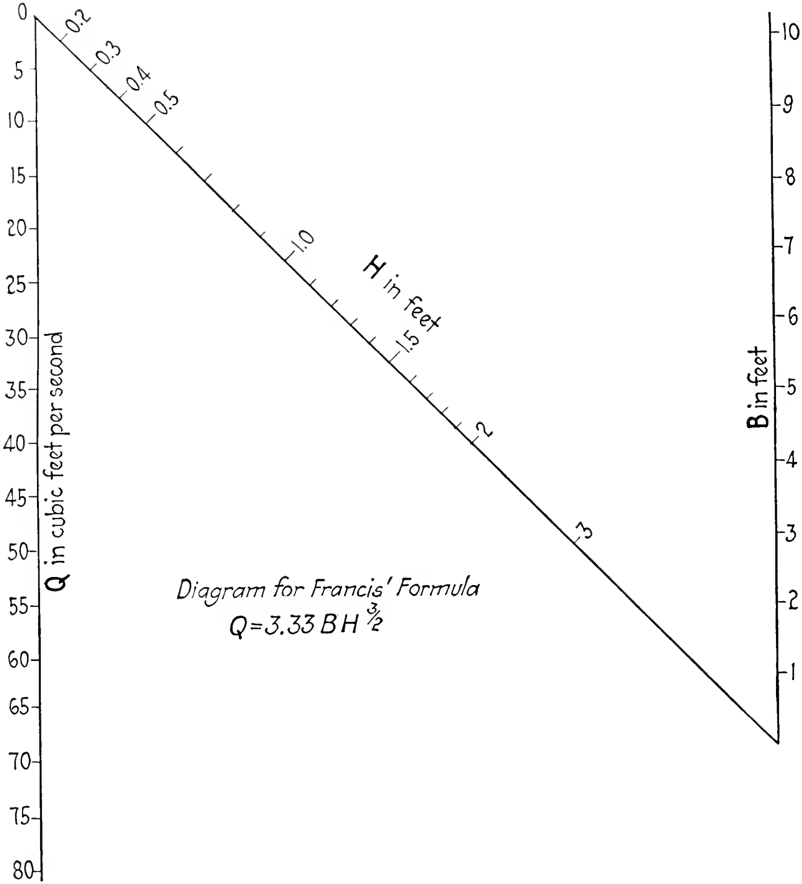

0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
Q in cubic feet per second
0.2
0.3
0.4
0.5
1.0
1.5
2
3
H in feet
Diagram for Francis' Formula
Q=3.33 B H 3/2
10
9
8
7
6
5
4
3
2
1
B in feet

FIG. 46.

The corresponding diagram is shown in Fig. 46 and the scales have been displaced by the method of the preceding section.

Example 28.—The formula

$$D = (.52)(14.7)(H)\left[\frac{1}{461+60} - \frac{1}{461+t}\right]$$

which gives the natural draft D in inches of water available from a chimney H feet high when the temperature of the gases is t degrees Fahrenheit, may be written

$$\begin{vmatrix} 0 & \left[0.014671 - \dfrac{7.644}{461+t}\right] & 1 \\ 1 & -D & 1 \\ \dfrac{1}{1+H} & 0 & 1 \end{vmatrix} = 0$$

In Fig. 47 the diagram is shown with

$$\mu_1 = 833 \qquad \mu_2 = 4 \qquad \delta = 9.16$$

and the scales are

$$x = 0 \qquad y = 833\left[0.014671 - \frac{7.644}{461+t}\right]$$

$$x = 1 \qquad y = -4D$$

$$x = \frac{(9.16)(833)\dfrac{1}{1+H}}{(833-4)\dfrac{1}{1+H}+4} = \frac{7{,}630}{833+4H} \qquad y = 0$$

The values of μ_1, μ_2 and δ were chosen in accordance with the methods of Article 4.

In Equation (8) of Article 11 if any pair of the functions f_i and g_i should have the form

$$f_i = \frac{a_1 z_i + b_1}{a_3 z_i + b_3} \qquad g_i = \frac{a_2 z_i + b_2}{a_3 z_i + b_3} \qquad i = 1, 2, 3$$

then the corresponding defining equations would still determine a straight scale, for upon elimination of z_i from a pair of such defining equations, there would result the straight line equation

$$(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y - (a_2b_1 - a_1b_2) = 0$$

In case all the functions in Equation (8) had the above form all three scales would be graduated on straight lines and no two would in general be parallel. All the equations so far discussed in this chapter are special cases of such forms. No essentially new type of diagram would result, however, in the more general case, for a projective transformation could always be found to change any such set of three given lines into a corresponding set either of all parallel lines or of two parallel and one non-parallel. For any set of three lines, no two of which are parallel, intersect either in one or in three points and it is only necessary to apply such a projective transformation to the configuration as will transform a point of intersection P with the coordinates (m, n) to infinity; then either the three or the two lines formerly intersecting at the point become parallel lines in the transformed position and one or the other of the cases already discussed would result.

To require that a point $P(m, n)$ be transformed to infinity it is only necessary that the general projective transformation whose equations are

$$x_1 = \frac{A_1x + B_1y + C_1}{A_3x + B_3y + C_3} \qquad y_1 = \frac{A_2x + B_2y + C_2}{A_3x + B_3y + C_3}$$

be chosen with coefficients A_3, B_3, C_3 satisfying the relation

$$A_3m + B_3n + C_3 = 0$$

(See Appendix B.) This projective transformation may in other respects be selected at will and would of course be made as simple as possible. As it has already been stated in Article 12, the effect of applying the projective transformation to the elements of the original determinant equation is to rearrange them by the laws of determinants.

In practice the general case of three non-parallel straight lines seldom occurs. When an equation does arise with the determinant form above discussed it is of course not necessary to transform it to a form with parallel scales as a few changes in the determinant will usually simplify the work of plotting the given scales.

14. Diagrams of Alignment with Curved Scales.—Before discussing equations or formulas in three variables which can be solved with diagrams of one or more curved scales, consider the general quadratic equation

$$z^2 + pz + q = 0$$

where the variables are z, p, and q. To set up the defining equations of a diagram, the equation may be reduced to a suitable determinant form, bearing in mind that no row shall contain more than one of the variables.

Select for the first row functions $f_1 = z$, and $g_1 = z^2$ and leave the third element position blank.

Then select $g_2 = p$ to combine with z and leave the other positions of the second row blank; there results

$$\begin{vmatrix} z & -z^2 & \\ & p & \end{vmatrix}$$

Now insert q as g_3 to avoid combinations with z and leave the remaining positions blank thus:

$$\begin{vmatrix} z & -z^2 & \\ & p & \\ & q & \end{vmatrix}$$

Then since the elements z and q and the product pz are terms of the original equation, there is sufficient guide to complete the determinant as follows:

$$\begin{vmatrix} z & -z^2 & 1 \\ 1 & p & 0 \\ 0 & q & 1 \end{vmatrix}$$

corresponding rows divided by its elements. But this would require the use of the reciprocals of p and q. To avoid using the reciprocals add the corresponding elements of columns one and three and form a new third column. Dividing the elements of the first row by $(1 + z)$ there results finally:

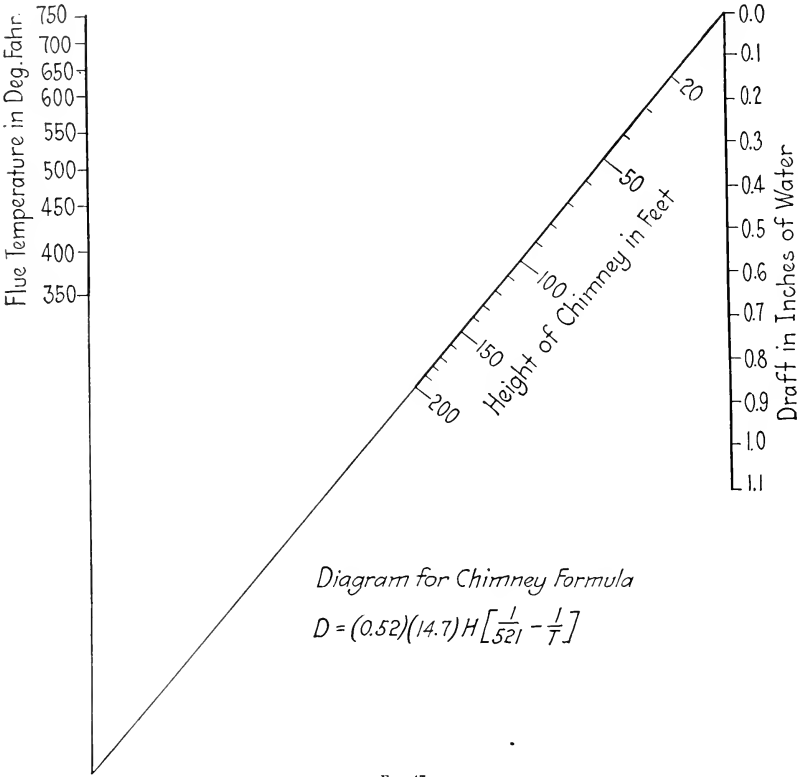


FIG. 47.

To put this first determinant equation into the reduced form (8), there are available all the elementary laws of determinants, and changes in sign of the determinant may be disregarded (not, however, changes in sign of the various elements unless sufficient to change the sign of the determinant). Since the second column is free from zero elements it might be moved into the third column position and then the

$$\begin{vmatrix} \dfrac{z}{1+z} & \dfrac{-z^2}{1+z} & 1 \\ 1 & p & 1 \\ 0 & q & 1 \end{vmatrix} = 0$$

Comparing this determinant with Equation (8) the equation of the three scales may for convenience be written as follows:

Example 29.

$$x = \frac{-z}{1+z} \qquad y = \frac{-z^2}{1+z}$$
$$x = -1 \qquad y = p$$
$$x = 0 \qquad y = q$$

The scale for z is then graduated on an hyperbola with the asymptote $x = -1$. Figure 48 is the diagram. No scale factors are used but the unit on the horizontal axis is taken 100 times that on the vertical axis, as this

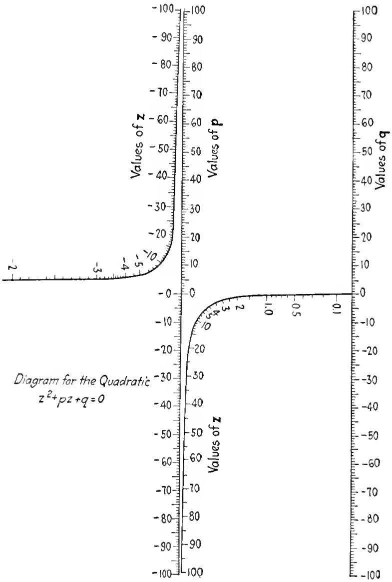


FIG. 48.

convenient device is always available. The roots are read at the points where the straight line through the given values of p and q on their respective scales, cuts this hyperbola graduated with z.

Consider now the process of obtaining a first determinant form for any given equation. The following steps outline a method of trial and error which will reduce most of the formulas of engineering:

First.—Select three or less functions of one variable which will deplete the formula of that variable and arrange them in any order as elements of a first row, leaving missing elements blank.

Second.—Arrange similar functions of a second variable as elements in a second row so that products required by the formula will result.

Third.—The remaining functions are elements of a third row and are inserted with regard to the resulting combinations.

Fourth.—Supply by inspection necessary constants (including zero) for missing elements, and rearrange terms *in the rows* until the expanded determinants check with the forms of the given equation.

When a first determinant is found it can readily be transformed to the type (8) by the laws of determinants. (See Appendix A.)

Example 30.—In Fig. 49 is shown a diagram for the cubic equation

$$z^3 + pz + q = 0$$

A first determinant form of the equation may be found by replacing $-z^2$ by $-z^3$ in the first determinant form for the quadratic equation. The reduced determinant form used for the present diagram was, however,

$$\begin{vmatrix} \frac{z-1}{z+1} & \frac{-z^3}{z+1} & 1 \\ 1 & p & 1 \\ -1 & q & 1 \end{vmatrix} = 0$$

The corresponding scale equations are

$$x = \frac{z-1}{z+1} \qquad y = \frac{-z^3}{z+1}$$
$$x = 1 \qquad y = p$$
$$x = -1 \qquad y = q$$

A simple form for the reduced determinant equation to which will correspond a diagram with two parallel straight scales and a curved scale is

$$\begin{vmatrix} 1 & g_1 & 1 \\ 0 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \qquad (18)$$

The quadratic equation above is an example. When this Equation (18) is expanded there results the form

$$g_2 + f_3(g_1 - g_2) - g_3 = 0 \qquad (19)$$

The functions f_3 and g_3 must of course not have the linear form discussed in the preceding section. The procedure for the introduction of scale factors in constructing corresponding scales is exactly as in the case of the equation of Article 13. If the first two defining equations are written

$$x = \delta \qquad y = \mu_1 g_1$$
$$x = 0 \qquad y = \mu_2 g_2$$

then the third pair of equations for the new curved scale will have the form

$$x = F_3 \qquad y = G_3$$

where F_3 and G_3 will be functions of f_3 and g_3 and involve the new constants δ, μ_1 and μ_2. Upon sub-

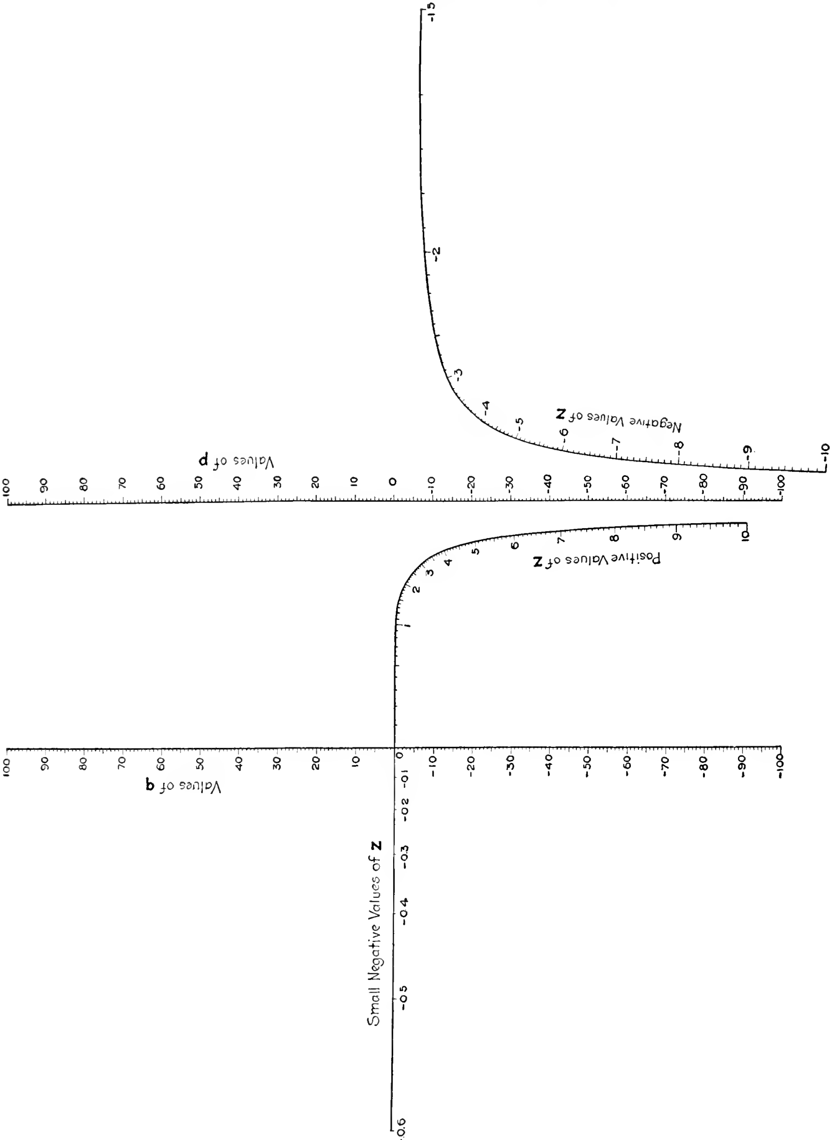


FIG. 49.—Diagram for the Cubic $z^3 + pz + q = 0$

stituting the value of g_1 from Equation (19) into the equation

$$\begin{vmatrix} \delta & \mu_1 g_1 & 1 \\ 0 & \mu_2 g_2 & 1 \\ F_3 & G_3 & 1 \end{vmatrix} = \delta\mu_2 g_2 + \mu_1 F_3 g_1 - \mu_2 g_2 F_3 - \delta G_3 = 0$$

there results the equation

$$g_2\left[\delta\mu_2 + \mu_1 F_3 \frac{(f_3 - 1)}{f_3} - \mu_2 F_3\right] + \frac{\mu_1 F_3 g_3}{f_3} - \delta G_3 = 0$$

which must be true for every g_2, hence the coefficient of g_2 and the term not involving g_2 must vanish identically and there follows, upon equating these expressions to zero

$$F_3 = \frac{\delta\mu_2 f_3}{\mu_2 f_3 - \mu_1(f_3 - 1)}, \quad G_3 = \frac{\mu_1\mu_2 g_3}{\mu_2 f_3 - \mu_1(f_3 - 1)} \qquad (20)$$

The above equations result also from the application to the original figure of the following projective transformation

$$x_1 = \frac{\delta\mu_2 x}{(\mu_2 - \mu_1)x + \mu_1}, \quad y_1 = \frac{\mu_1\mu_2 y}{(\mu_2 - \mu_1)x + \mu_1} \qquad (21)$$

and this projective transformation may be obtained by the methods explained in Appendix B.

There are given below two examples of equations arising in surveying practice for which diagrams are very useful and in both of which curved scales occur:

Example 31.—*Stadia Formula for Horizontal Distance.*—The distance H of a point from the instrument is

$$H = R - R\sin^2\alpha + c\cos\alpha$$

where: H = horizontal distance in feet
R = rod reading multiplied by 100
α = vertical angle
c = instrument constant, 0.85 to 1.15

With c taken as unity (which is sufficiently exact for most work) the formula may be given the first determinant form

$$\begin{vmatrix} 1 & H & 1 \\ 0 & R & 1 \\ 1 & \cos\alpha & \sin^2\alpha \end{vmatrix} = 0$$

When this determinant equation is given the form (8) by division of the last row by $\sin^2\alpha$ and the corresponding defining equations established, it is seen that because of the small values of the angle α (α seldom exceeds 30°) the diagram is impracticable. However, by adding the first and last columns to form a new third column and then dividing by the elements of the new third column, there results the form

$$\begin{vmatrix} \frac{1}{2} & \frac{H}{2} & 1 \\ 0 & R & 1 \\ \frac{1}{1+\sin^2\alpha} & \frac{\cos\alpha}{1+\sin^2\alpha} & 1 \end{vmatrix} = 0$$

which may finally be arranged as follows

$$\begin{vmatrix} 1 & H & 1 \\ 0 & 2R & 1 \\ \frac{2}{1+\sin^2\alpha} & \frac{2\cos\alpha}{1+\sin^2\alpha} & 1 \end{vmatrix} = 0$$

In this form which corresponds to Equation (18) the defining equations, with the needed scale factors $\delta = \frac{1}{21}$, $\mu_1 = \frac{1}{1{,}050}$, $\mu_2 = \frac{1}{2{,}000}$, may be written from Equations (21)

$$x = \frac{1}{21} \qquad y = \frac{H}{1{,}050}$$

$$x = 0 \qquad y = \frac{R}{1{,}000}$$

$$x = \frac{1}{1 + 20\sin^2\alpha} \qquad y = \frac{\cos\alpha}{50[1 + 20\sin^2\alpha]}$$

With the modulus unity equal to 20 inches the diagram of Fig. 50 was constructed. The scale for α is graduated on an hyperbola tangent at its vertex to the line $x = 1/21$. When $\alpha = 0$ the corresponding points on the scale are $x = 1$, $y = 1/50$. In order to bring small values of the angle α within the limits of the drawing the transformation

$$x_1 = x \qquad y_1 = x + y$$

was applied to the first diagram as defined. For actual practice in the office or field more suitable values of the constants may doubtless be chosen.

Example 32.—The formula for the vertical distance of a point above the level of the instrument is

$$V = \frac{R}{2}\sin 2\alpha + \sin\alpha$$

and this formula may be given the first determinant form

$$\begin{vmatrix} 1 & V & 0 \\ 0 & R & 1 \\ -1 & -\sin\alpha & \frac{\sin 2\alpha}{2} \end{vmatrix} = 0$$

By suitable transformations there results from this form the reduced determinant form of the equation

$$\begin{vmatrix} 1 & V & 1 \\ 0 & R & 1 \\ \frac{-2}{\sin 2\alpha - 2} & \frac{-2\sin\alpha}{\sin 2\alpha - 2} & 1 \end{vmatrix} = 0$$

Again using Equations (21) and the scale factors $\delta = \frac{2}{21}$, $\mu_1 = \frac{25}{10{,}000}$, $\mu_2 = \frac{1}{1{,}000}$, there result the defining equations

$$x = \frac{2}{21} \qquad y = \frac{25}{10{,}000}V$$

$$x = 0 \qquad y = \frac{1}{1{,}000}R$$

$$x = -\frac{4}{21}\cdot\frac{1}{2.5\sin 2\alpha - 2} \quad y = -\frac{5}{1{,}000}\cdot\frac{\sin\alpha}{2.5\sin 2\alpha - 2}$$

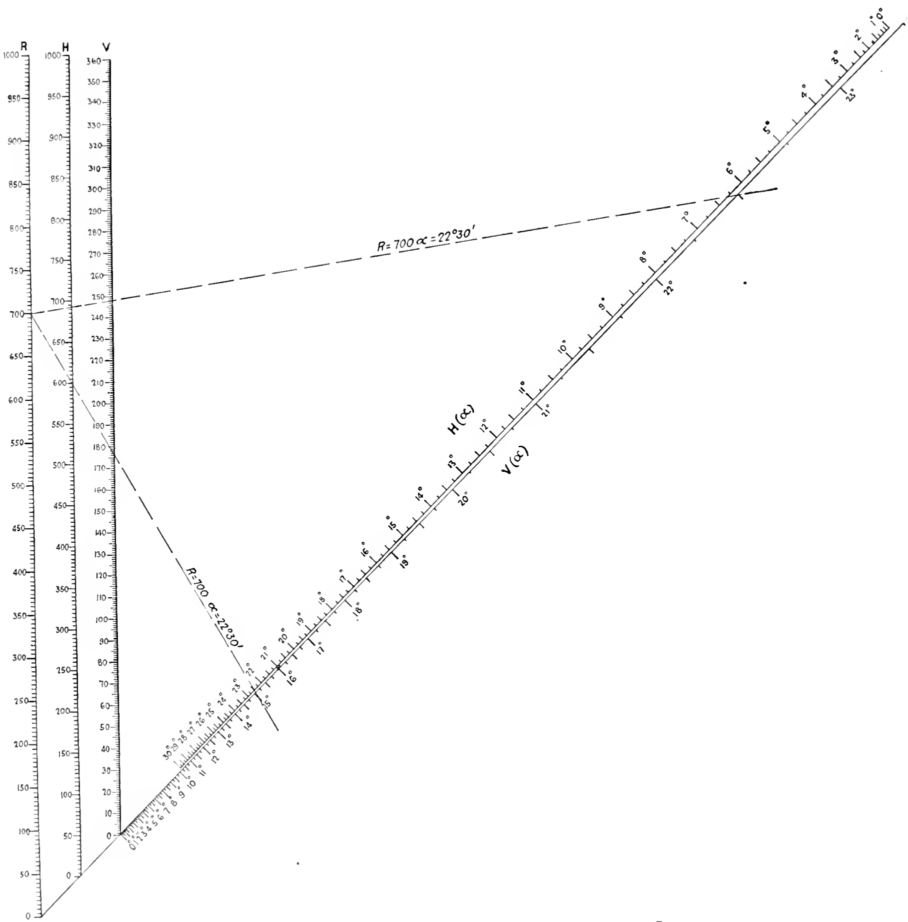


FIG. 50.—Diagram for the Stadia Formulas $H = R - R\sin^2\alpha + \cos\alpha$ and $V = \frac{R}{2}\sin 2\alpha + \sin\alpha$.

The values of α are graduated on a quartic curve. The diagram is combined with the diagram for the horizontal distance worked out above as the two quantities H and V are always computed together. The reader will readily see that the same transformation was necessarily applied to both parts of the diagram to improve the arrangement of the values of the vertical angle. Figure 50 shows the combined diagram.

It is not of course necessary that the two straight scales be parallel when there is but one curved scale. Below is given an example where the two straight scales are graduated on the axes of coordinates.

Example 33.—The formula for the mean hydraulic radius of trapezoidal sections of canals may be written

$$R = \frac{h(b + h \cot \phi)}{b + 2h\sqrt{1 + \cot^2 \phi}}$$

Where R = mean hydraulic radius
h = depth of water
b = width of canal bottom
ϕ = angle which the side slope makes with the horizontal.

Write $\frac{R}{h} = r$ and $\frac{h}{b} = k$ and a final determinant equation is

$$\begin{vmatrix} 0 & K & 1 \\ \frac{r-1}{2r} & 0 & 1 \\ \frac{1}{2} - \sec \phi & -\tan \phi & 1 \end{vmatrix} = 0$$

The defining equations can be written without the use of scale factors as follows

$$x = 0 \qquad y = K$$
$$x = \frac{r-1}{2r} \qquad y = 0$$
$$x = \tfrac{1}{2} - \sec \phi \qquad y = -\tan \phi$$

The values of the angle ϕ are graduated on an equilateral hyperbola crossing the X axis at $x = -\frac{1}{2}$. The diagram is shown in Fig. 51.

When h is given there is of course not much disadvantage in computing R from the value of $\frac{R}{h}$.

If it were required to use scale factors to establish a diagram for this equation above which has actually the form

$$\begin{vmatrix} 0 & g_1 & 1 \\ f_2 & 0 & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \tag{22}$$

or expanded, the form

$$g_1 f_3 + f_2 g_3 - f_2 g_1 = 0 \tag{23}$$

then the projective transformations developed in treating Equation (18) are available. The equations were

$$x_1 = \frac{\delta\mu_2 x}{(\mu_2 - \mu_1)x + \mu_1} \qquad y_1 = \frac{\mu_1\mu_2 y}{(\mu_2 - \mu_1)x + \mu_1} \tag{21}$$

Then the new defining equations corresponding to Equation (22) would be

$$x = 0 \qquad y = \mu_2 g_1$$
$$x = \frac{\delta\mu_2 f_2}{(\mu_2 - \mu_1)f_2 + \mu_1} \qquad y = 0$$
$$x = \frac{\delta\mu_2 f_3}{(\mu_2 - \mu_1)f_3 + \mu_1} \qquad y = \frac{\mu_1\mu_2 g_3}{(\mu_2 - \mu_1)f_3 + \mu_1}$$

Another simple reduced determinant equation for which there are two parallel straight scales and a curved scale is

$$\begin{vmatrix} -1 & g_1 & 1 \\ 1 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \tag{24}$$

The expanded form of this equation is

$$(g_1 + g_2) - f_3(g_1 - g_2) - 2g_3 = 0 \tag{25}$$

The defining equations for the curved scale will undergo a change should scale factors be introduced in the equations of the straight scales. Suppose that it is desired to have the two parallel scales at equal distances δ from the Y axis and to use the scale factors μ_1 and μ_2 respectively. The first defining equations as before have the form

$$x = -\delta \qquad y = \mu_1 g_1$$
$$x = \delta \qquad y = \mu_2 g_2$$

and the third defining equations must be assumed to have the form

$$x = F_3 \qquad y = G_3$$

where F_3 and G_3 are to be functions of f_3 and g_3 alone and will involve the constants δ, μ_1 and μ_2. The reduced determinant form of the equation will then become

$$\begin{vmatrix} -\delta & \mu_1 g_1 & 1 \\ \delta & \mu_2 g_2 & 1 \\ F_3 & G_3 & 1 \end{vmatrix} = 0$$

and upon expanding there results

$$\delta(\mu_1 g_1 + \mu_2 g_2) - F_3(\mu_1 g_1 - \mu_2 g_2) - 2\delta G_3 = 0$$

But from Equation (25)

$$g_1 = \frac{g_2 - 2g_3 + f_3 g_2}{f_3 - 1}$$

which substituted in the equation above yields

$$g_2\left[\delta\mu_2 + \delta\mu_1\frac{1 + f_3}{f_3 - 1} - F_3\left\{\mu_1\frac{1 + f_3}{f_3 - 1} - \mu_2\right\}\right] - 2\left[\frac{\delta\mu_1 g_3 - F_3\mu_1 g_3}{f_3 - 1} - \delta G_3\right] = 0$$

and this equation must be true for every value of g_2.

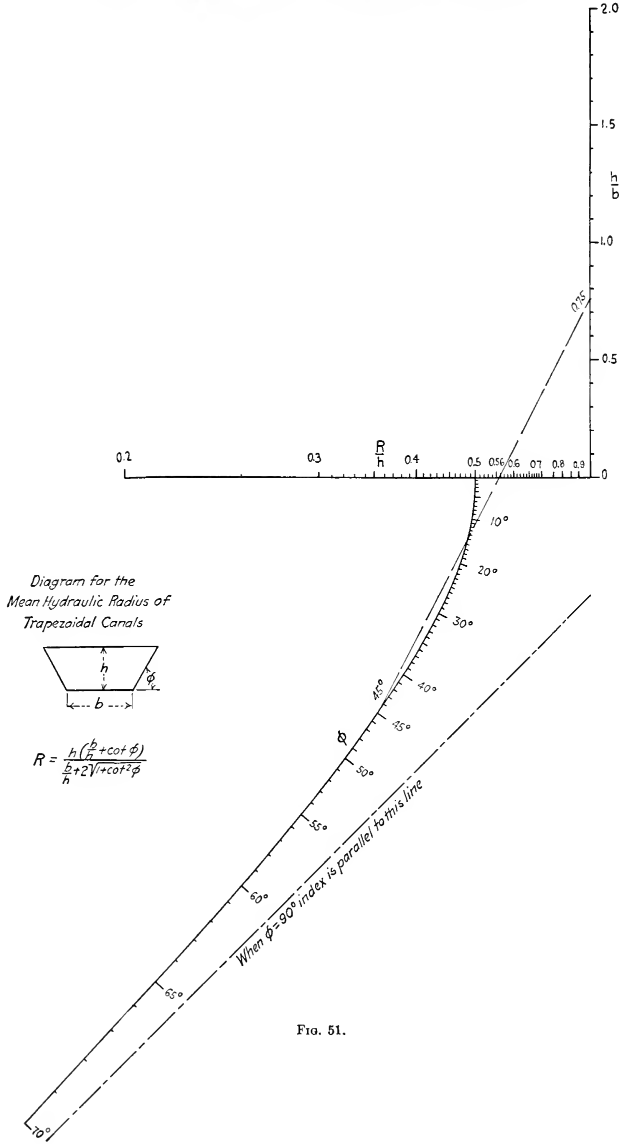


Diagram for the Mean Hydraulic Radius of Trapezoidal Canals
$R = \dfrac{h\left(\frac{b}{h}+\cot\phi\right)}{\frac{b}{h}+2\sqrt{1+\cot^2\phi}}$
h
b
φ
R/h
h/b
0.2
0.3
0.4
0.5
0.56
0.6
0.7
0.8
0.9
0
0.5
0.75
1.0
1.5
2.0
10°
20°
30°
40°
45°
50°
55°
60°
65°
70°
When φ = 90° index is parallel to this line

Fig. 51.

Therefore the coefficient of g_2 and the term not involving g_2 must vanish identically, that is

$$F_3 = \delta\frac{(\mu_1 + \mu_2)f_3 + (\mu_1 - \mu_2)}{(\mu_1 - \mu_2)f_3 + (\mu_1 + \mu_2)}$$

$$G_3 = \frac{2\mu_1\mu_2 g_3}{(\mu_1 - \mu_2)f_3 + (\mu_1 + \mu_2)}$$

and the changed form of the equations of the curved scale of the diagram become

$$x = \delta\frac{(\mu_1 + \mu_2)f_3 + (\mu_1 - \mu_2)}{(\mu_1 - \mu_2)f_3 + (\mu_1 + \mu_2)} \qquad (26)$$

$$y = \frac{2\mu_1\mu_2 g_3}{(\mu_1 - \mu_2)f_3 + (\mu_1 + \mu_2)}$$

The above equations are very important for the construction of diagrams discussed in the succeeding sections of this book. They are the result of the application of the projective transformation

$$x_1 = \delta\frac{(\mu_1 + \mu_2)x + (\mu_1 - \mu_2)}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)} \qquad (27)$$

$$y_1 = \frac{2\mu_1\mu_2 y}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$

to the points of the figure as originally defined.

It is seldom that the functions in an engineering formula similar to Equation (8) are of such general form that more than one curved scale results in the diagram and indeed no rule can be given for the introduction of scale factors when the defining equations are of the most general form. It is impossible for example to introduce different pairs of values of δ and μ in the first two defining equations for if no restriction were placed on the nature of the functions f_1, f_2, g_1, g_2, the first two curves originally defined might intersect in one or more points and to use the scale factors δ_1 and μ_1; and δ_2 and μ_2 would generally demand that the same points of intersection of the curves supporting the two original scales must move in different directions at the same time and take new positions. It is necessary, consequently, to leave to the reader the introduction of desirable scale factors in those cases of Equation (8) not already treated. It will be necessary to take advantage of the particular form of the individual equation in hand and to use the general methods here developed. A thorough understanding of the use of the projective transformation which is developed in Appendix B is very helpful.

15. Diagrams of Alignment with One Fixed Point.—Equation (16) of Article 13 may be written

$$f_1 - f_3 g_2 = 0 \qquad (28)$$

and given the determinant form

$$\begin{vmatrix} f_1 & g_2 & 1 \\ f_3 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \qquad (29)$$

Then the three pairs of equations

$$x = f_1 \qquad y = g_2$$
$$x = f_3 \qquad y = 1$$
$$x = 0 \qquad y = 0$$

define respectively: All the points of the plane, all the points of the line $y = 1$ (a straight function scale), and the origin. A diagram may be designed on suitable cross-section paper with abscissas as values of f_1 and ordinates as values of g_2 and inscribed with corresponding values of the variables z_1 and z_2, and with a scale of the function f_3 on the line $y = 1$. Then the values of z which constitute a solution of Equation (29) are collinear. Since the index always passes through the origin it may be scratched on a piece of celluloid pivoted at that point.

Example 34.—The formula of Francis

$$q = 3.33BH^{3/2}$$

yields a diagram of the above type and the defining equations are conveniently:

$$x = \frac{q}{10} \qquad y = B$$
$$x = 3.33H^{3} \qquad y = 10$$
$$x = 0 \qquad y = 0$$

See Fig. 52.

Whenever the functions f_1 and g_2 are linear functions of the variables z_1 and z_2 respectively ordinary cross-section paper may be used quickly to establish the desired diagram. It is of course optional which function f_3 or g_2 is used in the first row of the determinant.

By using logarithmic cross-section paper, equations of the form

$$f_1 - g_2^{z_3} = 0$$

may readily be solved for a limited range of the variables involved to almost any desired degree of accuracy. Passing to logarithms

$$\log f_1 - z_3 \log g_2 = 0$$

and with the defining equations from the determinant of Equation (29) there results

$$x = \log f_1 \qquad y = \log g_2$$
$$x = z_3 \qquad y = 1$$
$$x = 0 \qquad y = 0$$

There is an ordinary scale on the line $y = 1$ and the logarithmic cross-section paper is inscribed with values of z_1 and z_2.

Example 35.—Frequently in thermodynamics the equation

$$PV^n = C$$

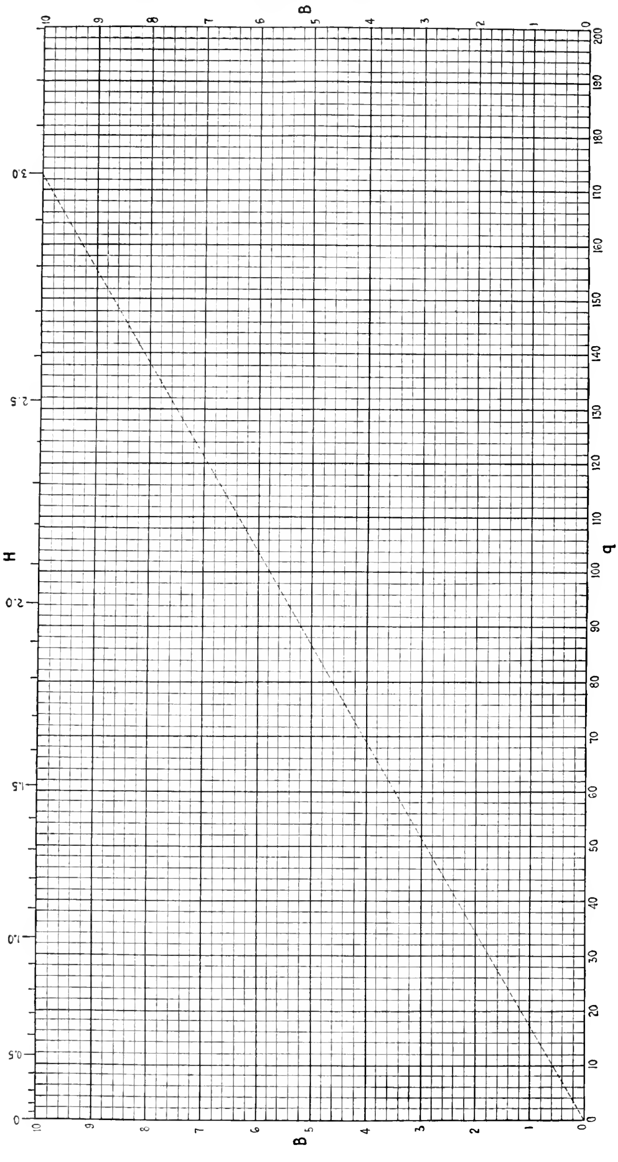


Fig. 52.—Diagram for Francis Weir Formula $q = 3.33\,BH^{3/2}$.

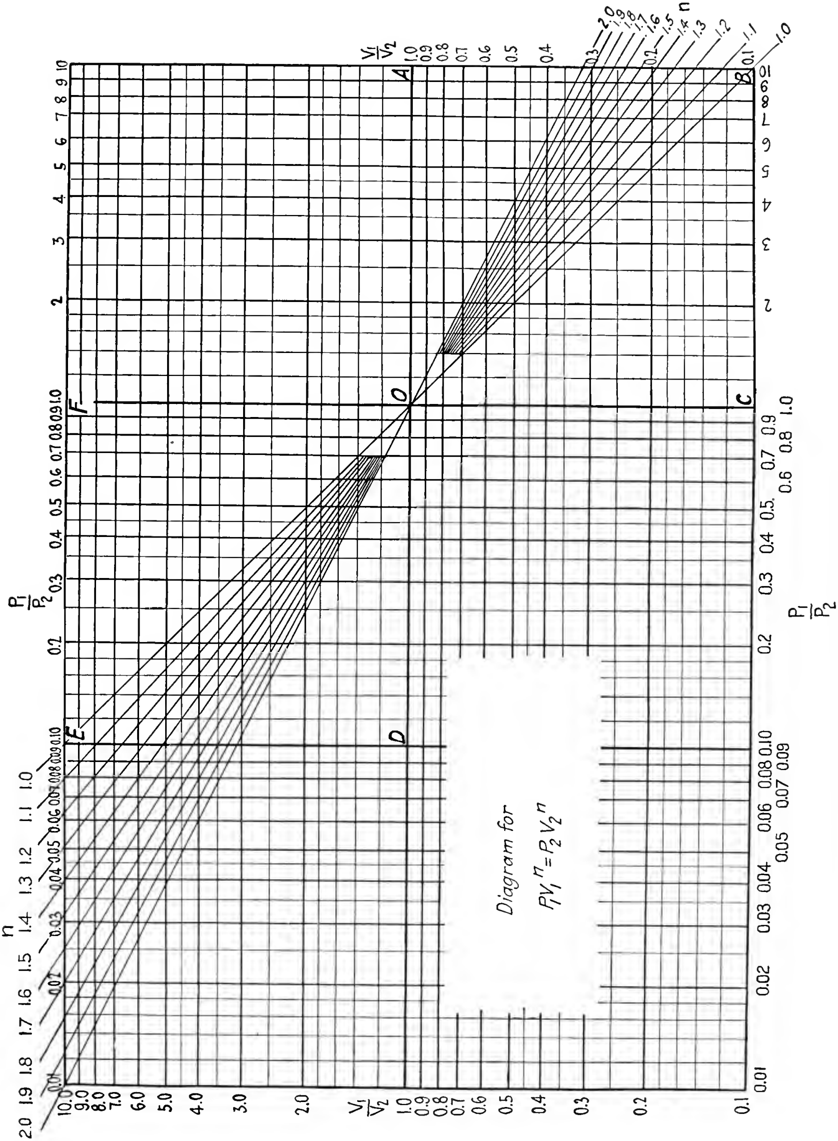


FIG. 53.

arises, and with a given set of values of C and n it is desired to find an indefinite number of closely determined points on the curve plotted with P as ordinates and V as abscissas. It is usually desired to find additional pairs of values P_2 and V_2 to satisfy the equation

$$P_1V_1^n = P_2V_2^n$$

The defining equations of a suitable diagram of the type under discussion are then

$$x = \log \Delta P \qquad y = \log \Delta V$$
$$x = -n \qquad y = 1$$
$$x = 0 \qquad y = 0$$

and the diagram is shown in Fig. 53.

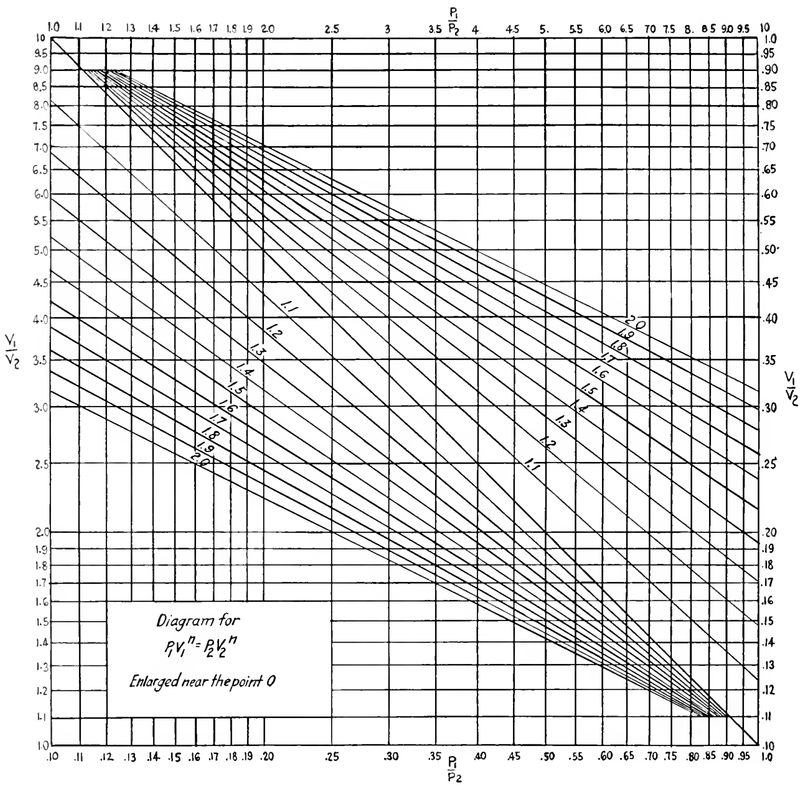


FIG. 53a.

where P_1 and V_1 have been determined. This equation may be written

$$[\log P_1 - \log P_2] + n[\log V_1 - \log V_2] = 0$$

or

$$\log \Delta P + n \log \Delta V = 0$$

While the diagram consists essentially of the logarithmic cross-section paper with the scale for n and the origin clearly marked upon it, in this figure the various positions of the index have been drawn in as straight lines for n. This is allowable since the equation

$$\log \Delta P + n \log \Delta V = 0$$

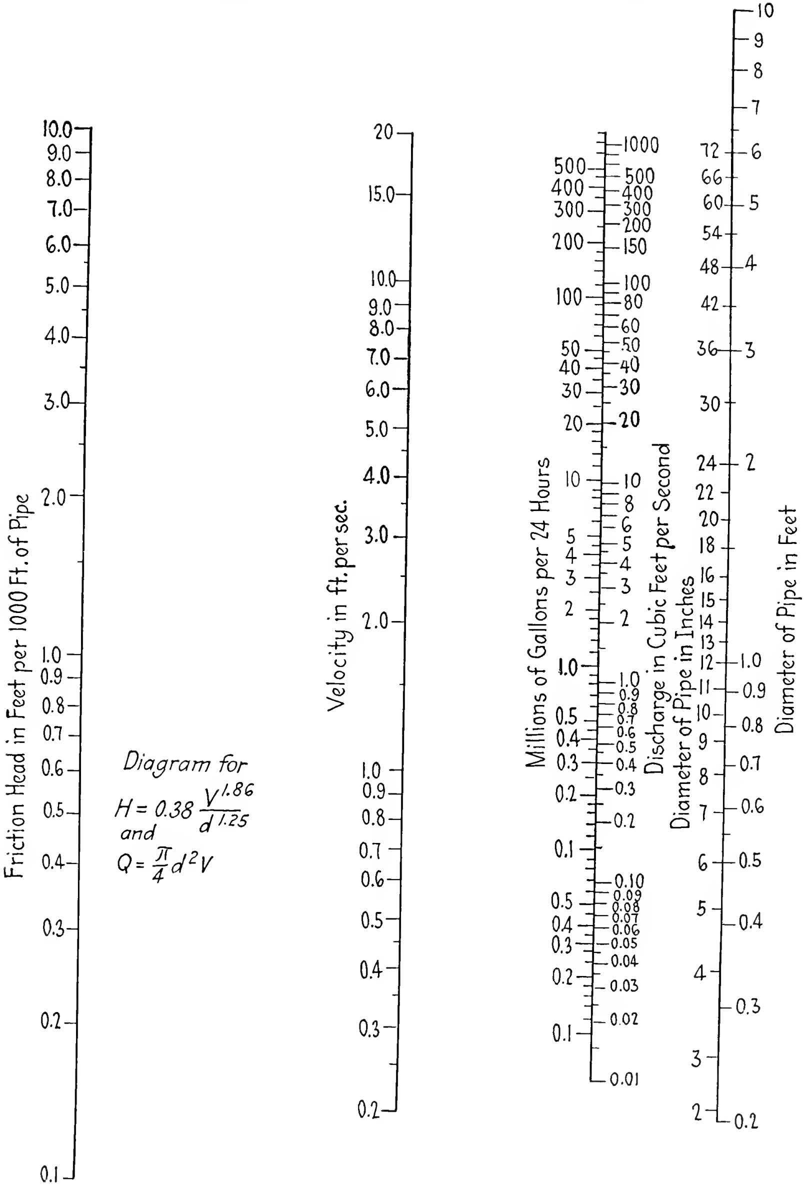

Friction Head in Feet per 1000 Ft. of Pipe
Velocity in ft. per sec.
Millions of Gallons per 24 Hours
Discharge in Cubic Feet per Second
Diameter of Pipe in Inches
Diameter of Pipe in Feet
Diagram for
$H = 0.38 \frac{V^{1.86}}{d^{1.25}}$
and
$Q = \frac{\pi}{4} d^2 V$

Fig. 54.

is also in the special form of equation (5) Article 8 which was shown to yield a family of radial straight lines through the origin. Since the range of numbers for n is small, the useful area of Fig. 53 is rather limited. The two squares $OABC$ and $ODEF$ nearest the origin have been superimposed and drawn to a larger scale in Fig. 53a. This gives a more convenient and accurate diagram which is in a form suitable for rapid plotting of the $PV^n = C$ curve. The inscribed values of $\frac{P_1}{P_2}$ and $\frac{V_1}{V_2}$ found on all four sides of the logarithmic cross-section paper of Fig. 53a apply only to the half of the diagram between them and the line $n = 1$.

Equation (29) when $f_1 = c$, $g_2 = b$, and $f_3 = a$, will determine a multiplication diagram for the product $ab = c$. If the various positions of the index are drawn through the origin for products of integers and tenths, a convenient multiplication diagram results for small numbers. By suitable choice of scale factors, diagrams for products of special ranges of numbers may be prepared.

A useful extension of this diagram results if the radial lines through the origin showing the positions of the index are drawn on transparent celluloid, together with the scale $x = a$, $y = 1$ and the resulting "index" arranged to permit translation along the X axis over the cross-section sheet defining the lines $x = c$, $y = b$. For then by determining a first product $c_1 = a_1b_1$ on the X axis with the "index" axes coincident with the axes of coordinates (on the cross-section paper beneath) there may be added to c_1 the product $c_2 = a_2b_2$ by translating the "index" to the point c_1 on the X axis and then reading $c = c_1 + c_2$ on the X axis at the foot of the ordinate through the intersection of the "index" line a_2 and the second multiplicand line corresponding to b_2. (See Problem 12, Chapter III.)

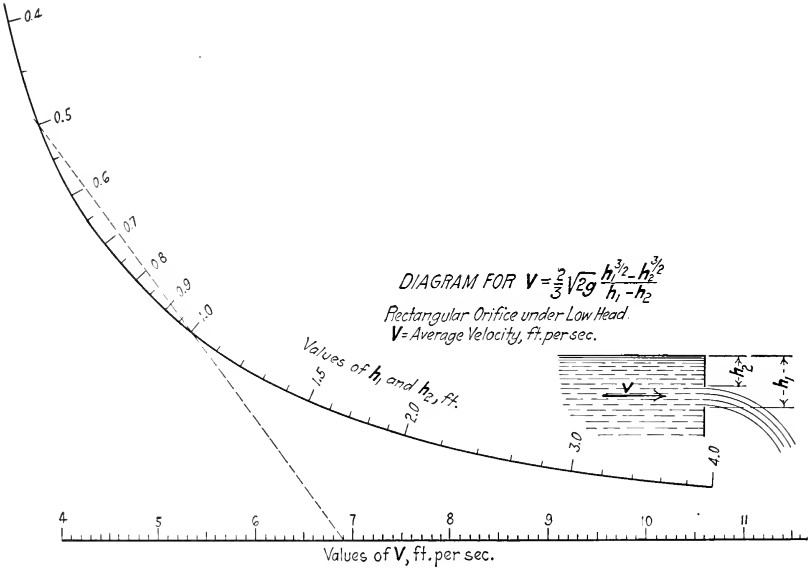


FIG. 55.

Problem 1.—The formula of Francis

$$q = 3.33BH^{3/2}$$

is of the form of Equation (12). Construct a diagram for this formula using suitable scale factors. Another method of treating this same formula is given in Section 13 and still another in Section 15.

Problem 2.—The formulas

$$H = 0.38\frac{V^{1.86}}{d^{1.25}} \text{ and } Q = \frac{\pi}{4}d^2V$$

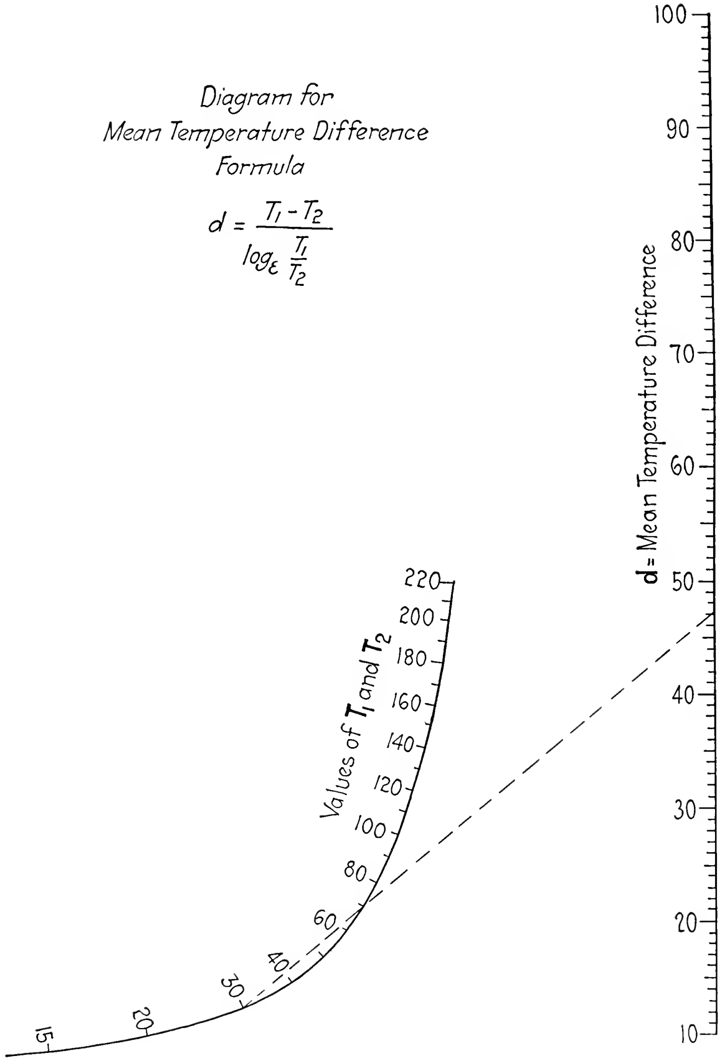


Diagram for
Mean Temperature Difference
Formula
$d = \frac{T_1 - T_2}{\log_\varepsilon \frac{T_1}{T_2}}$
Values of T_1 and T_2
15
20
30
40
60
80
100
120
140
160
180
200
220
d = Mean Temperature Difference
10
20
30
40
50
60
70
80
90
100

FIG. 56.

were combined in Fig. 30 and four systems of lines resulted. In Fig. 54 are shown four straight line scales for the same equations. Write the equations necessary to describe the method of constructing Fig. 54.

Problem 3.—Show how Francis' formula in Example 27 is given the determinant form there used by virtue of Equations (17) and write three other possible determinant forms.

Problem 4.—Test the accuracy of Figs. 48 and 49 by setting up quadratic and cubic equations with known roots.

Problem 5.—Show how the determinant form of Example 30 was obtained by the laws of determinants. (See Appendix A.)

Problem 6.—Draw a diagram for the above cubic equation based on a reduced determinant form of the equation analogous to that used for the quadratic equation.

Problem 7.—A formula for the approximate area of a circular segment of radius R and with height H is

$$A = \frac{4H^2}{3}\sqrt{\frac{2R}{H} - 0.608}$$

It is possible to reduce this formula to the form of Equation (18) above. Construct a diagram for it.

Problem 8.—Develop equations for introducing scale factors in Equation (22) by the methods used in the other cases of the present chapter.

Problem 9.—Write the set of equations showing the analysis of Figs. 55 and 56.

Problem 10.—Show that the formula for the mean hydraulic radius of trapezoidal sections of Example 33 may be represented by a diagram with two straight scales and a third scale inscribed upon a circle. The necessary reduced determinant form may be derived from the determinant equation of Example 33 by the laws of determinants.

Problem 11.—Show that the mean hydraulic radius of the circular segment of Fig. 41 has the form

$$M.H.R. = \tfrac{1}{2}rf(K)$$

where $f(K)$ means as before a function of the ratio $\frac{H}{R}$, and design a suitable diagram for the mean hydraulic radius of circular sewers flowing at any depth.

Problem 12.—Frequently in civil engineering "end areas" are determined by a formula

$$A = \tfrac{1}{2}[(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) + \ . \ . \ . \ + (x_1 - x_n)(y_1 + y_n)]$$

when x_i represents distances from a center line, and y_i cuts and fills at the corresponding points; construct a diagram with a sliding index to compute A for values of x_i and y_i varying by tenths up to 20 feet.

Problem 13.—Devise a combined diagram to handle the following relation in thermodynamics

$$\frac{T_1}{T_2} = \left(\frac{P_1}{P_2}\right)^{\frac{n-1}{n}} = \left(\frac{V_1}{V_2}\right)^{1-n}$$

Problem 14.—Construct a diagram of three parallel straight scales for the expression of Problem 14 of Chapter II.

Problem 15.—The tractive resistance R in pounds of an automobile of weight W pounds when moving at a speed of V miles per hour is given by Prof. E. H. Lockwood as

$$R = 15 + .015W + .075V^2$$

Construct a diagram for this formula.

Problem 16.—Construct a diagram consisting of four parallel straight line scales upon which the collineation of four points will serve to solve the two equations

$$P_a = \sqrt[3]{P_1^2P_2}$$
$$P_b = \sqrt[3]{P_1P_2^2}$$

as used in determining the ideal intercooler pressures in a three stage air compressor. P_1 = initial pressure, P_2 = final pressure, P_a = first intercooler pressure, P_b = second intercooler pressure, all in pounds per square inch absolute.

Problem 17.—Using the methods of Article 15 investigate the necessary scale factors to establish a working diagram for finding the present value of one dollar at interest rates of 2 to 8 per cent and for periods of 5 to 20 years by the formula

$$v^n = (1 + i)^{-n}$$

where v^n = present value of 1 due in n periods or years
i = interest rate (decimal)
n = number of interest periods or years.

CHAPTER IV

ALIGNMENT DIAGRAMS FOR FORMULAS IN MORE THAN THREE VARIABLES

16. Binary Function Scales and Curve Nets.— Suppose that in the XY plane there are plotted two systems of curves,

$$\phi_1(xy) = z_1 \qquad \phi_2(xy) = z_2 \qquad (N)$$

as shown in Fig. 57.

Through every point P of the plane will pass a curve of each system inscribed with its corresponding value of z. This configuration of curves will be called the *curve net* N_{12} for z_1 and z_2. A line perpendicular to OX is seen to cut out an indefinite number of pairs of values of z_1 and z_2. Every point M of OX may thus be regarded as supplied with all the pairs of values of z_1 and z_2 which correspond to the curves intersecting on the line PM. These value pairs cannot all be written at the point M but are nevertheless definitely attached to it. Furthermore, given the value of z_1 there is but one[1] corresponding value of z_2 to be found upon PM.

If now every point M on OX is regarded as supplied in this way with its values of z_1 and z_2, the line OX becomes a certain kind of scale. Each length OM determines uniquely a line MP on which lies a certain set of values z_1z_2.

Let $OM = x_1$ and consider the line $x_1 = x_1$ and the curves

$$\phi_1(xy) = z_1 \qquad \phi_2(xy) = z_2$$

Eliminating x and y from these three equations yields an equation in z_1, z_2 and x which may be written

$$f(z_1z_2) = x_1$$

All the values of z_1z_2 which satisfy the above equations belong to the point M.

Conversely, given a value of z_1 (or z_2) and the point M (which is equivalent to assuming the value of x_1), there is in general but one value of z_2 (or z_1) which will satisfy the last equation. It is thus convenient to define the configuration of Fig. 57 as a *binary function scale* for the function f_{12} on OX which is called the *support*.

[1] If the line PM intersects the curve corresponding to z_1 in n points there will of course correspond n values of z_2, etc.

Similarly eliminating x from the equations (N) leads to the result

$$g(z_1z_2) = y$$

The curve net of Fig. 57 thus completely determines also a binary function scale for g_{12} on OY.

Frequently a pair of functions f_{12} and g_{12} occur in an equation of four variables for which a diagram is to be constructed, and when the equation is put into

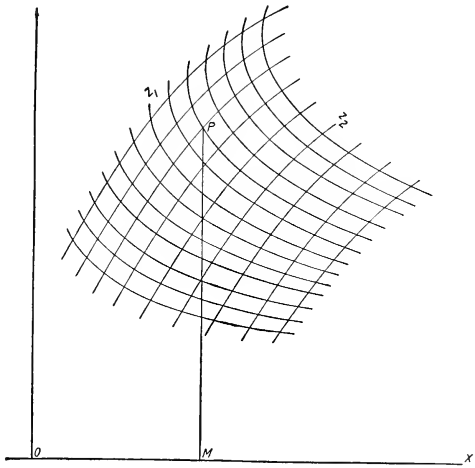


FIG. 57.

the determinant form analogous to Equation (8) it is necessary to interpret the defining equations

$$x = f_{12} \qquad y = g_{12}$$

It is evident from the foregoing that these two equations define a curve net. It is merely necessary to eliminate z_2 and z_1 successively and there is obtained again

$$\phi_1(xy) = z_1 \qquad \phi_2(xy) = z_2$$

Since the only necessary equations for a binary scale on the X axis are

$$x = f_{12} \quad \text{and} \quad y = 0$$

they are called the *defining equations for the binary scale*. In constructing a curve net N_{12} for this scale it is seen that either of the functions ϕ_1 or ϕ_2 may be arbitrarily chosen. When, however, one function, say ϕ_1, is chosen the other function ϕ_2 is determined by eliminating z_1 from the equations

$$\phi_1(xy) = z_1 \quad \text{and} \quad f(z_1z_2) = x$$

Obviously ϕ_1 cannot be a function of x alone. Examples below will show, however, that a suitable choice of the arbitrary function aids in the solution.

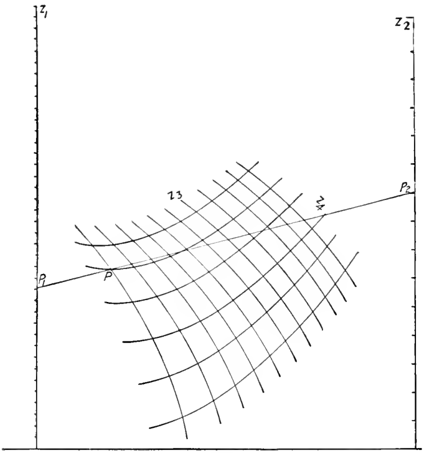


FIG. 58.

The binary scale is really then a special case of a curve net resulting when either of the functions f_{12} or g_{12} defining a curve net reduces to zero or any constant. When either z_1 or z_2 only is absent from f_{12} or from g_{12} one set of curves in the corresponding net will be a system of parallel straight lines. (For another special case see Article 23 of Chapter VI.) When a binary scale has been established on either axis or upon a line parallel to either axis it is obvious that the necessary curve net may be moved by translation parallel to the other axis provided that the straight line support remains fixed. Obviously then a new origin of the axes of coordinates may thus be chosen for plotting the necessary curve net and this method is sometimes of much advantage in improving the plan of the diagram.

17. Collinear Diagrams with Two Parallel Scales and One Curve Net.—Consider now the equation in four variables $z_i(i = 1, 2, 3, 4)$, which may be given the determinant form

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_{34} & g_{34} & 1 \end{vmatrix} = 0 \qquad (30)$$

A simple case arises when this reduced determinant equation may be written analogous to Equation (18) of Chapter III:

$$\begin{vmatrix} 1 & g_1 & 1 \\ 0 & g_2 & 1 \\ f_{34} & g_{34} & 1 \end{vmatrix} = 0 \qquad (31)$$

When this equation is expanded there results

$$g_2 + f_{34}(g_1 - g_2) - g_{34} = 0 \qquad (32)$$

Assume temporarily that the scale factors are unity and there results a set of defining equations from Equation (31):

$$x = 1 \qquad y = g_1$$
$$x = 0 \qquad y = g_2$$
$$x = f_{34} \qquad y = g_{34}$$

and the last two equations define a curve net. It is thus necessary to study a collinear nomogram or diagram of alignment consisting of two parallel straight scales and a set of points defined by an inscribed curve net. To each point of the curve net corresponds a pair of values z_1z_2 attached to the two curves passing through that point. The equations of the curve net are readily written by eliminating z_4 and z_3 successively from the last two equations and they become

$$\phi_3(xy) = z_3 \qquad \phi_4(xy) = z_4$$

The resulting configuration is shown schematically in Fig. 58.

Given three values of z_i, the diagram of Fig. 58 constitutes a complete graphic solution for the unknown value of z. Suppose that z_4 is unknown: The line P_1P_2 cuts then the curve z_3 in the point P through which passes a curve marked z_4. The proof that this value of z_4 is the value sought is left to the reader.

For certain equations in four variables there is thus realized an important type of collinear diagram. To be solvable by such a diagram an equation must be reducible to the form (31). Obviously the parallel scales may be placed at a distance δ and the scale factors μ_1 and μ_2 employed if the equations of the curve net are determined from the third pair of defining equations as modified by the Equations (21)

of Chapter III. There results then for the defining equations

$$
\begin{aligned}
&x = \delta && y = \mu_1 g_1 \\
&x = 0 && y = \mu_2 g_2 \qquad (33) \\
&x = \frac{\delta\mu_2 f_{34}}{\mu_2 f_{34} - \mu_1(f_{34} - 1)} && y = \frac{\mu_1\mu_2 g_{34}}{\mu_2 f_{34} - \mu_1(f_{34} - 1)}
\end{aligned}
$$

The choice of the constants δ, μ_1 and μ_2 should of course be made not only with the first two scales in view but also with the resultant changes in the curve net fully in mind. No plotting should be undertaken until a thorough study of the equations has been made in order to obtain the desired range of values of the variables involved and at the same time to reduce as far as possible the required computation for plotting the curve net.

Example 36.—A very good illustrative example is afforded by the complete cubic equation

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0$$

which may be given the determinant form

$$\begin{vmatrix} 1 & a_2 & 1 \\ 0 & a_1 & 1 \\ \dfrac{z}{z^2 + z} & \dfrac{-(z^3 + a_3)}{z^2 + z} & 1 \end{vmatrix} = 0$$

Whence if $\delta = 10$, $\mu_1 = \mu_2 = 1$,

$$
\begin{aligned}
&x = 10 && y = a_2 \\
&x = 0 && y = a_1 \\
&x = \frac{10z}{z^2 + z} && y = -\frac{z^3 + a_3}{z^2 + z}
\end{aligned}
$$

are the defining equations for the diagram which is shown in Fig. 59. In plotting the curve net for the variables z and a_3 the z lines parallel to the Y axis are plotted first and then it is observed that the successive a_3 curves determine regular scales on each z line with a new scale factor for each. It is only necessary to plot the curves for the values of a_3 equal to -10, 0, and 10 successively to determine completely the system of curves. The scale factor on each z line is seen to be $\dfrac{1}{z^2 + z}$. In the diagram the dotted line shows the position of a straight edge set to solve the equation $z^3 + 4z^2 - 4z + 0.5 = 0$. The straight edge is set from $a_1 = +4$ to $a_2 = -4$ and gives the value of $z = 0.69$ at its intersection with the curve $a_3 = 0.5$.

Another simple case of Equation (30) which results in the same form of diagram is

$$\begin{vmatrix} -1 & g_1 & 1 \\ 1 & g_2 & 1 \\ f_{34} & g_{34} & 1 \end{vmatrix} = 0 \qquad (34)$$

The expanded equation has the form

$$2g_{34} + f_{34}(g_1 - g_2) - (g_1 + g_2) = 0 \qquad (35)$$

and the defining equations with the scale factors determined by the aid of Equations (25) and (26) for the analogous case of three variables, are

$$
\begin{aligned}
&x = -\delta && y = \mu_1 g_1 \\
&x = \delta && y = \mu_2 g_2 \qquad (36) \\
&x = \delta\frac{(\mu_1 + \mu_2)f_{34} + (\mu_1 - \mu_2)}{(\mu_1 - \mu_2)f_{34} + (\mu_1 + \mu_2)} && \\
& && y = \frac{2\mu_1\mu_2 g_{34}}{(\mu_1 - \mu_2)f_{34} + (\mu_1 + \mu_2)}
\end{aligned}
$$

The presence of the constants δ and μ in the third pair of equations allows control to some extent of the disposition of the resulting curve net. Whenever the scales for the first two variables extend in opposite directions in the diagram it is desirable to apply a transformation as in Section 12 of Chapter III. This is done in the following illustrative examples.

From the last pair of defining equations in (33) and (36) it is seen that whenever z_3 or z_4 is absent from f_{34} there results a system of straight lines parallel to the Y axis and they are determined by a scale on the X axis most conveniently. Whenever f_{34} or g_{34} is zero (or when f_{34} is constant) there result the defining equations of a binary scale on the Y axis or on the X axis (or on the line x = constant) respectively. Another special case occurs which leads to a curved binary scale and is discussed in Chapter VI.

Example 37.—As an illustrative example of Equation (34) consider Kutter's formula for the flow of water in open channels,

$$V = \frac{41.6603 + \dfrac{1.81132}{n} + \dfrac{0.00281}{S}}{1 + \left[41.6603 + \dfrac{0.00281}{S}\right]\dfrac{n}{\sqrt{R}}}\sqrt{RS}$$

Where V = velocity in feet per second
S = tangent of inclination of surface
R = mean hydraulic radius
n = Kutter's coefficient of channel bottom.

The above formula may be modified by setting

$$S = \frac{1}{1{,}000}$$

outside the radical.[1] There results

$$V = \frac{44.4703 + \dfrac{1.81132}{n}}{1 + \dfrac{44.4703\,n}{\sqrt{R}}}\sqrt{RS}$$

which, if $44.4703 = a$, and $1.81132 = b$, reduces to

$$\begin{vmatrix} -1 & V & 0 \\ 0 & -\sqrt{S} & 1 \\ -n(\sqrt{R} + an) & 0 & (an + b)R \end{vmatrix} = 0$$

[1] This substitution is known as Flynn's modification of Kutter's formula.

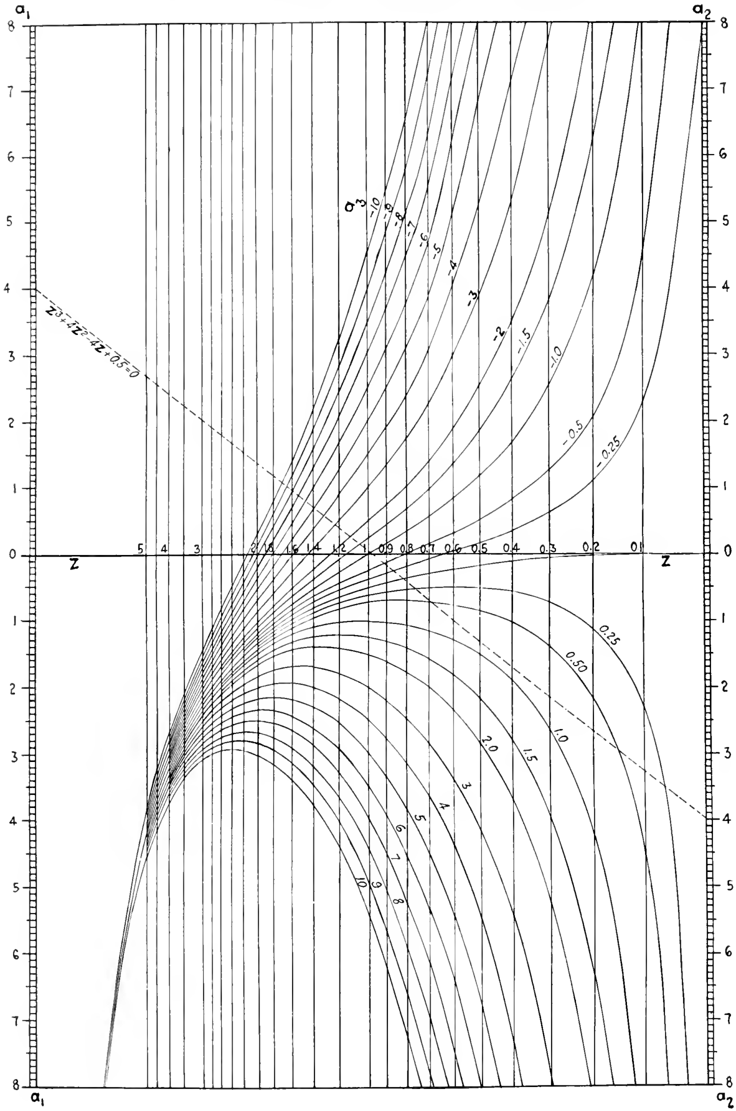


Fig. 59.—Diagram for the Complete Cubic Equation $z^3 + a_1z^2 + a_2z + a_3 = 0$.

as a first determinant form. The reduced determinant form is then found to be

$$\begin{vmatrix} -1 & V & 1 \\ 1 & -\sqrt{S} & 1 \\ \dfrac{(an+b)R - n(\sqrt{R}+an)}{(an+b)R + n(\sqrt{R}+an)} & 0 & 1 \end{vmatrix} = 0$$

The defining equations written from Equations (36) above are

$$x = -\delta \qquad y = \mu_1 V$$
$$x = \delta \qquad y = -\mu_2\sqrt{S}$$
$$x = \delta\frac{\mu_1(an+b)R - \mu_2 n(\sqrt{R}+an)}{\mu_1(an+b)R + \mu_2 n(\sqrt{R}+an)} \qquad y = 0$$

For convenience then in plotting there may be chosen

$$\delta = 10, \qquad \mu_1 = 0.8 \qquad \mu_2 = 80.0$$

whence the scale equations:

$$x = -10 \qquad y = 0.8V$$
$$x = 10 \qquad y = -80\sqrt{S}$$
$$x = 10\frac{(an+b)R - 100n(\sqrt{R}+an)}{(an+b)R + 100n(\sqrt{R}+an)} \qquad y = 0$$

Since g_{34} is here zero there is a binary scale on the X axis. One system of curves in the net defining the binary scale may well be chosen as the parallel lines

$$y = 2\sqrt{R}$$

and there follows upon eliminating R the cubic curves for n

$$x = 10\frac{(an+b)y^2 - 200n(y+2an)}{(an+b)y^2 + 200n(y+2an)}$$

All these cubics pass through the point $x = -10$, $y = 0$ and are asymptotic to the vertical line $x = 10$. (See Fig. 60.)

The V and S scales would naturally lie in opposite directions from the X axis but to secure a better disposition of these scales and thus reduce the size of the sheet, they have been moved by using the projective transformation

$$x_1 = x \qquad y_1 = x + y + 10$$

which moves all points along their ordinates a distance equal to the abscissa plus 10. Thus the line $y = 0$ becomes the line $y = x + 10$ which is the line MN in the diagram. From the nature of the binary scale, however, there is no need of transforming the curve net for the variables n and R and this has not been done in the figure. The points on the binary scale are simply transferred by the parallel vertical straight lines from the X axis to the diagonal which thus becomes the new support.

Example 38.—Another example of Equation (34) is afforded by Bazin's formula for the flow of water in open channels which is

$$V = \frac{87}{0.552 + \dfrac{m}{\sqrt{R}}}\sqrt{RS}$$

where V, R, and S have the same meaning as above and m is Bazin's coefficient of bottom condition.

The first determinant form of the equation may be written

$$\begin{vmatrix} 1 & V & 0 \\ 0 & -\sqrt{S} & 1 \\ (0.552R+m) & 0 & 87R \end{vmatrix} = 0$$

and the reduced form of the equation is then

$$\begin{vmatrix} -1 & V & 1 \\ 1 & -\sqrt{S} & 1 \\ \dfrac{87R - 0.552\sqrt{R} - m}{87R + 0.552\sqrt{R} + m} & 0 & 1 \end{vmatrix} = 0$$

The corresponding scale equations are

$$x = -\delta \qquad y = \mu_1 V$$
$$x = \delta \qquad y = -\mu_2\sqrt{S}$$
$$x = \delta\frac{\mu_1 87R - \mu_2(0.552\sqrt{R}+m)}{\mu_1 87R + \mu_2(0.552\sqrt{R}+m)} \qquad y = 0$$

There is again a binary scale on the X axis which is determined by setting

$\delta = 10$, $\mu_1 = 0.8$ $\mu_2 = 80$ as above and also $y = 2\sqrt{R}$

whence $$x = 10\frac{87y^2 - 200(0.552y+m)}{87y^2 + 200(0.552y+m)}$$

The m curves of the corresponding net are six cubics for Bazin's six values of m. These cubics have a singular point at $x = -10$, $y = 0$ and are asymptotic to the line $x = 10$. Figure 61 shows the finished diagram originally plotted with a modulus of one inch. The S scale and the binary scale have been transformed as in the preceding example.

Equation (31) is no simpler than the essentially equivalent form

$$\begin{vmatrix} f_{34} & g_{34} & 1 \\ f_1 & 1 & 1 \\ f_2 & 0 & 1 \end{vmatrix} = 0$$

which has the expanded form:

$$f_{34} + g_{34}(f_2 - f_1) - f_2 = 0$$

and for which the corresponding diagram will consist of two horizontal (instead of vertical) parallel scales and a curve net. To introduce scale factors into the corresponding defining equations there are available Equations (21) of Chapter III. With the obvious changes in the role of the respective coordinates x and y there results from these equations

$$x_1 = \frac{\mu_1\mu_2 x}{(\mu_2-\mu_1)y + \mu_1} \qquad y_1 = \frac{\delta\mu_2 y}{(\mu_2-\mu_1)y + \mu_1}$$

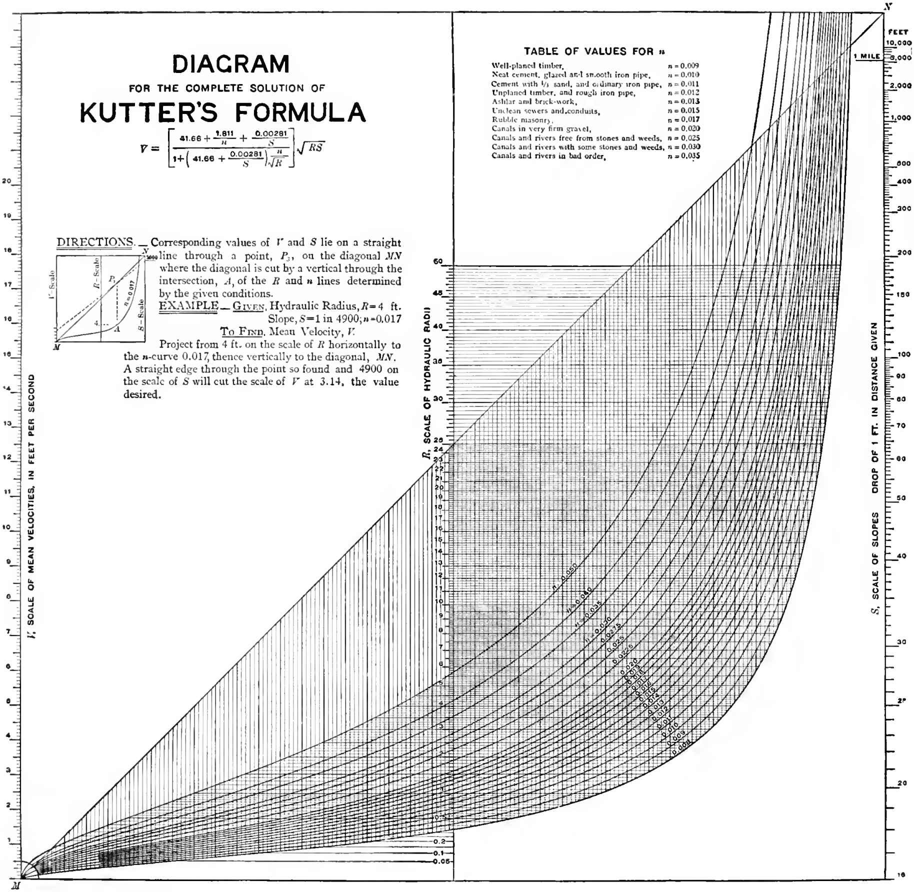


DIAGRAM
FOR THE COMPLETE SOLUTION OF
KUTTER'S FORMULA
$V=\left[\dfrac{41.66+\dfrac{1.811}{n}+\dfrac{0.00281}{S}}{1+\left(41.66+\dfrac{0.00281}{S}\right)\dfrac{n}{\sqrt{R}}}\right]\sqrt{RS}$
TABLE OF VALUES FOR n
Well-planed timber, n = 0.009
Neat cement, glazed and smooth iron pipe, n = 0.010
Cement with 1/3 sand, and ordinary iron pipe, n = 0.011
Unplaned timber, and rough iron pipe, n = 0.012
Ashlar and brick-work, n = 0.013
Unclean sewers and conduits, n = 0.015
Rubble masonry, n = 0.017
Canals in very firm gravel, n = 0.020
Canals and rivers free from stones and weeds, n = 0.025
Canals and rivers with some stones and weeds, n = 0.030
Canals and rivers in bad order, n = 0.035
DIRECTIONS.—Corresponding values of V and S lie on a straight line through a point, P_3, on the diagonal MN where the diagonal is cut by a vertical through the intersection, A, of the R and n lines determined by the given conditions.
EXAMPLE—GIVEN, Hydraulic Radius, $R=4$ ft. Slope, $S=1$ in 4900; $n=0.017$
TO FIND, Mean Velocity, V.
Project from 4 ft. on the scale of R horizontally to the n-curve 0.017, thence vertically to the diagonal, MN. A straight edge through the point so found and 4900 on the scale of S will cut the scale of V at 3.14, the value desired.
V-Scale
R-Scale
S-Scale
P_3
A
n=0.017
V, SCALE OF MEAN VELOCITIES, IN FEET PER SECOND
R, SCALE OF HYDRAULIC RADII
S, SCALE OF SLOPES DROP OF 1 FT. IN DISTANCE GIVEN
FEET
1 MILE
M
N

FIG. 60.

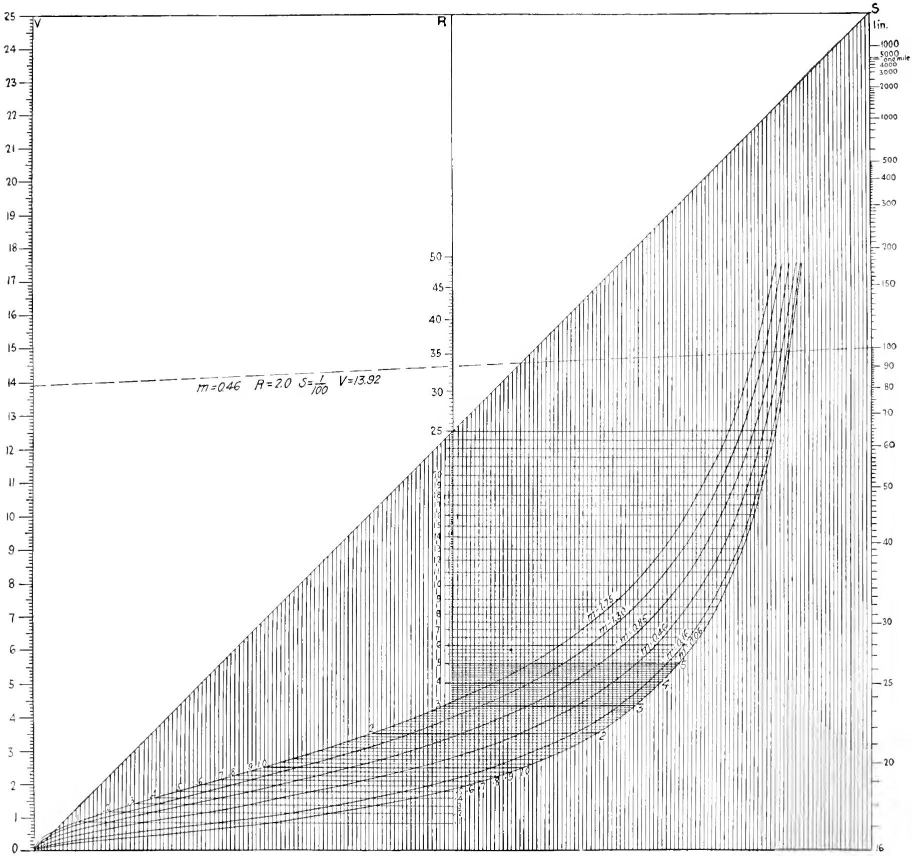


FIG. 61.—Diagram for Bazin's Formula $V = \dfrac{87}{0.552 + \dfrac{m}{\sqrt{R}}}\sqrt{RS}$.

Example 39.—One form of the fundamental formula for bond calculations is

$$\frac{A}{C} = v^n + \frac{g}{C}\left(\frac{1 - v^n}{i}\right)$$

where A = purchase price
g = nominal interest rate
i = effective or yield rate of interest
C = redemption price
n = term of bond in years

$$v = \frac{1}{1 + i}$$

This bond formula has the reduced determinant form:

$$\begin{vmatrix} \bar{R} & 0 & 1 \\ g' & 1 & 1 \\ \dfrac{v^n}{1 - A_{\overline{n}|}} & \dfrac{-A_{\overline{n}|}}{1 - A_{\overline{n}|}} & 1 \end{vmatrix} = 0$$

in which for convenience the symbol $A_{\overline{n}|}$ is used for $\dfrac{1 - v^n}{i}$ and in which $g' = \dfrac{g}{C}$ and $\bar{R} = \dfrac{A}{C}$. This is an excellent example of the form of Equation (31) above. By letting $\mu_2 = 1$ and $\mu_1 = \mu$ and substituting successively in the modified Equations (21) above the respective pairs of elements from the determinant there result the defining equations

$$x = \bar{R} \qquad y = 0$$
$$x = \mu g' \qquad y = \delta$$
$$x = \frac{\mu v^n}{\mu - A_{\overline{n}|}} \qquad y = -\frac{\delta A_{\overline{n}|}}{\mu - A_{\overline{n}|}}$$

The curve net for i and n defined by the last pair of equations may best be plotted as follows:

The ratio $$\frac{x}{y} = -\frac{\mu}{\delta}\frac{v^n}{A_{\overline{n}|}} = -\frac{\mu}{\delta}\cdot\frac{iv^n}{1 - v^n}$$

from which $$v^n = \frac{\delta x}{\delta x - i\mu y}$$

and when this value of v^n is substituted in the second equation

$$y = -\frac{\delta A_{\overline{n}|}}{\mu - A_{\overline{n}|}} = \frac{-\delta(1 - v^n)}{i\mu - 1 + v^n}$$

there results

$$y = \frac{\delta(x - 1)}{i\mu - 1}$$

which defines a pencil of lines through $x = 1$, $y = 0$ as the i-lines. The equation of the n-curves may be shown to be

$$\left(\frac{x + 2y - 1}{y}\right)^n = \frac{1 - y}{x}$$

if both μ and δ are taken unity but it is not necessary to attempt to plot from this equation. Instead resume the equation

$$\frac{y}{x} = -\delta\frac{(1 - v^n)}{\mu i v^n}$$

the quantity $\dfrac{1 - v^n}{iv^n}$ is tabulated in standard works on bonds, life insurance, etc. and is designated $S_{\overline{n}|}$ and rewritten

$$S_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$$

From these double entry tables when i is constant the values of $S_{\overline{n}|}$ vary for n only, thus the equation

$$\frac{y}{x} = -\frac{\delta}{\mu}S_{\overline{n}|}$$

will determine a pencil of lines through the origin varying for values of n. These lines intersect the corresponding i-line in points necessarily on the respective n-curves. Thus the n-curves may easily be plotted.

The useful range of values of the ratio $\dfrac{A}{C}$ is from say $^{75}\!/_{100}$ to $^{120}\!/_{100}$ and to be effective in actual bond calculations this ratio must be readable to the nearest thousandth or tenth of a per cent, consequently if the scale should show one per cent as one-half inch the effective portion would be $22\frac{1}{2}$ inches long and unity would be represented by 50 inches. The choice of μ and δ must then be made and it is obvious that if δ is greater than unity the line $y = \delta$ on which is to be shown the g' scale will be not only off any drawing of dimension less than 50 inches vertically but also (since g' will never be much greater than $\frac{1}{10}$) unless μ is large g' will be too close to the Y axis to appear on any reasonably sized drawing where unity is 50 inches. The remedy for both these troubles is to choose oblique axes with a very acute angle. When this is done and with $\delta = \frac{9}{10}$ and $\mu = 2$ there results the completed drawing shown in Fig. 62. It is observed that the nominal interest scale is inscribed g and not g'. This is because the normal case is redemption at par and then g' reduces to g the normal rate. With $\mu = 2$, one per cent on the g scale is represented by one inch. The auxiliary net of lines for the segregation from the binary scale of the ratio $\dfrac{A}{C}$ into the purchase price A and the redemption price C is effected by the equations

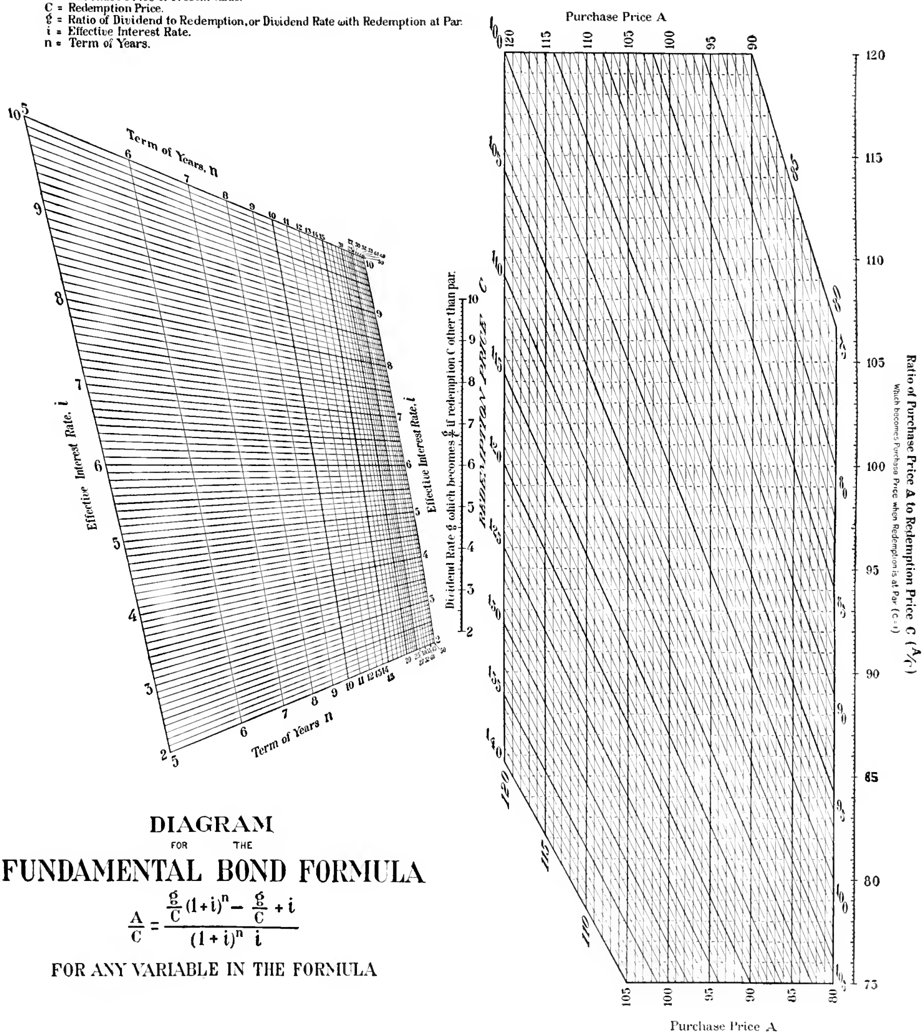
$$x = \frac{A}{C} \qquad y = mA \qquad m = \text{constant}$$

The choice of $\mu = 2$ is dictated by the behavior of the n curves for a reasonable range of useful terms n and was determined only after several trials.

The examples here worked out are special cases of Equations (31) and (34) which are both special cases of the more general Equation (30) which equation would in general require two curved scales and a curve

LEGEND

A = Purchase Price or Present Value.
C = Redemption Price.
g = Ratio of Dividend to Redemption, or Dividend Rate with Redemption at Par.
i = Effective Interest Rate.
n = Term of Years.



DIAGRAM

FOR THE

FUNDAMENTAL BOND FORMULA

$$\frac{A}{C} = \frac{\frac{g}{C}(1+i)^n - \frac{g}{C} + i}{(1+i)^n\ i}$$

FOR ANY VARIABLE IN THE FORMULA

Fig. 62.

net for its diagram. Examples encountered in practice seldom require such a type of diagram but treatment of the scales by some projective transformation would doubtless be needed for any such example.

It is to be observed that whenever one set of curves in a curve net becomes a system of straight lines, then the plotting of the second set of curves can often be simplified by finding indirectly their intersections with this plotted line system. Such was essentially the method used in Examples 36 and 39.

More generally, when one set of curves of a curve net has been plotted, the second set can be plotted by determining indirectly points of intersection with individual curves of the plotted first set. To do this hold constant in either of the defining equations the value of the variable parameter corresponding to given curve, while the second variable corresponding to the desired set is allowed to vary and draw the resulting lines parallel to one of the axes. There will thus be determined on the plotted curve a series of points of intersection corresponding to successive values of the second variable. These points for constant values of the second variable on successive curves will lie on a curve of the second set. In particular if, as above explained, the first set of curves is a system of straight lines, then the curves of the second system can always be found by plotting corresponding points of intersection of this first system of lines with the system of lines parallel to either one of the axes. In Example 39 an auxiliary set of lines through the origin was used to advantage instead of the parallel lines determined by a defining equation.

18. Collinear Diagrams with Three Curve Nets.— These diagrams and indeed diagrams with two curve nets are largely of theoretic interest but there are special cases of practical value.

Consider first an equation of six variables in the determinant form

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{34} & g_{34} & 1 \\ f_{56} & g_{56} & 1 \end{vmatrix} = 0 \qquad (37)$$

By setting

$$x = f_{ij} \qquad y = g_{ij} \qquad \begin{matrix} i = 1, 3, 5 \\ j = 2, 4, 6 \end{matrix}$$

there are obtained three curve nets constituting a collinear diagram for this equation. The key to the solution of the diagram is obvious from the schematic Fig. 63. Should the active range of the variables involved determine curve nets which unduly overlap or confuse the diagram, some device such as different colors will be needed to make the drawing of practical value. In most cases that occur in practice the curve nets reduce to binary scales and seldom are there more than two.

Example 40.—As an illustrative example consider the equation for the angular distance z of a celestial body east or west of the meridian from the north point.

$$\cos\frac{1}{2}z = \sqrt{\frac{\cos S \cos (S - p)}{\cos L \cos h}}$$

where $L =$ the latitude of observer

$p =$ the polar distance of the object

$h =$ the altitude of the object

$S = \frac{1}{2}(h + L + p)$

This equation may be solved by a diagram with two binary scales, but since z must usually be determined

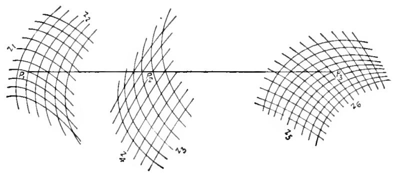


FIG. 63.

at least to the nearest 30 seconds no diagram of any practical value can be drawn small enough to reproduce here successfully. The variables are z, S, $(S - p)$, L, and h. After squaring both sides of the equation it may be written in the reduced determinant form

$$\begin{vmatrix} 0 & \cos S \cos (S - p) & 1 \\ 1 & -\cos^2\dfrac{z}{2} & 1 \\ \dfrac{\cos L \cos h}{1 + \cos L \cos h} & 0 & 1 \end{vmatrix} = 0$$

There may be written in a manner analogous to Equation (14) of Chapter III, the defining equations

$$x = 0 \qquad y = \mu_1 \cos S \cos (S - p)$$

$$x = \delta \qquad y = -\mu_2 \cos^2\frac{z}{2}$$

$$x = \delta\frac{\mu_1 \cos L \cos h}{\mu_2 \cos L \cos h + \mu_2} \qquad y = 0$$

The first two equations define a binary scale on the Y axis and the variables S and $(S - p)$ may be separated with the curve net

$$x = \cos S \qquad y = \mu_1 x \cos (S - p)$$

which gives two systems of straight lines. The second equation pair defines the scale of length μ_2 measured downward on the line $x = \delta$. The third equation

pair determines a binary scale on the X axis and is constructed with the simple curve net

$$x = \delta \frac{-\mu_1 \cos Ly}{\mu_2 - \mu_1 \cos Ly} \qquad y = -\cos h$$

The L curves are then equilateral hyperbolas passing through the origin and with asymptotes parallel to the coordinate axes.

Problem 1.—Consider the cubic equation

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0$$

from the point of view of Equation (34) and write the defining equations of a diagram with parallel scales for a_2 and a_3.

Problem 2.—Consider the above cubic from the point of view of Equation (31) and write the equations of the curve net resulting if the parallel scales are for the variables a_2 and a_3.

Problem 3.—Assuming that the first two pairs of defining equations of Equation (34) are written

$$x = -\delta_1 \qquad y = \mu_1 g_1$$
$$x = \delta_2 \qquad y = \mu_2 g_2$$

show that the third pair of defining equations are

$$x = \delta \frac{(\delta_2\mu_1 + \delta_1\mu_2)f_{34} + (\delta_2\mu_1 - \delta_1\mu_2)}{(\mu_1 - \mu_2)f_{34} + (\mu_1 + \mu_2)}$$

$$y = \frac{2\mu_1\mu_2 g_{34}}{(\mu_1 - \mu_2)f_{34} + (\mu_1 + \mu_2)}$$

Problem 4.—The area of a trapezoidal section of an irrigation canal is given by the formula

$$A = hb + h^2 \cot \phi$$

where the symbols are used in the same sense as in Example 33, Chapter III. Construct a diagram for the formula which shall have two straight parallel scales for A and b and a curve net for h and ϕ consisting respectively of lines parallel to the Y axis and hyperbolas passing through the origin and tangent to the X axis at that point.

Problem 5.—Professor C. H. Forsyth has given[1] a formula for the premium or discount per unit on a bond if the "amortization factor" accumulates at a rate of interest r which is different from the effective or yield rate of the bond i. If k denotes this premium or discount then with the notation of Example 39 and redemption at face value or par the formula is

$$k = \frac{g - i}{\dfrac{1}{S^r_{\overline{n}|}} + i}$$

where the changed symbol $\dfrac{1}{S^r_{\overline{n}|}}$ denotes that this tabulated quantity $\dfrac{1}{S_{\overline{n}|}}$ is to be here taken at the rate r. Show that this formula is a special case of Equation (37) with five variables and with a corresponding diagram which consists of a straight line (cross-section) net for g and k, an ordinary scale for i on the Y axis and a binary scale on the line $x = -1$ for $\dfrac{1}{S^r_{\overline{n}|}}$ and that the segregation of the n and r lines in the binary scale net can be obtained by setting $x = r - 1$ and plotting the n curves by determining points on the r line corresponding to changes in n for constant r.

Problem 6.—In the above problem show that the equations for the curve net for g and k can also have the equations

$$x = \frac{-\delta\mu_2 k}{(\mu_2 - \mu_1)k + \mu_1} \qquad y = \frac{\mu_1\mu_2 g}{(\mu_2 - \mu_1)k + \mu_1}$$

if scale factors δ, μ_1, μ_2 are introduced by Equation (21) of Chapter III and that consequently an ordinary cross-section net for g and k results when $\mu_1 = \mu_2$.

Problem 7.—The so-called premium formula for bond valuation is with the usual notation

$$k = (g - i)\, A^i_{\overline{n}|}$$

where $A^i_{\overline{n}|}$ indicates that $A_{\overline{n}|}$ is to be evaluated at the rate i. When the bond is bought at a discount k is negative. Compare this equation with that of Example 39 and discuss the advantages, if any, for design of the corresponding diagram.

Problem 8.—Show that the equation for the n-curves in the curve net of the diagram of the above equation are given by the equation: $\left(\dfrac{x + y}{y}\right)^n = \dfrac{1 + y}{1 + y - x}$.

Problem 9.—If all interests are payable m times a year and the amortization factor accumulates at a nominal rate r then the premium formula of Professor Forsyth becomes

$$k = \frac{g - j}{\dfrac{m}{S_{\overline{mn}|}} + j}$$

where $S_{\overline{mn}|}$ is to be evaluated for mn periods at rate $\dfrac{r}{m}$ and where j is the nominal rate to be realized. Show how this equation in the five variables k, g, j, r, n, can also be diagrammatically represented for values of m from $m = 1$ to $m = 4$.

Problem 10.—Show in Example 36 how the cubic curves for a_3 could have been plotted by first plotting a system of lines parallel to the X axis which would intersect a given z-line parallel to the Y axis in points corresponding to values of a_3.

[1] *Bulletin*, Am. Math. Soc., vol. XXVII, p. 451.

CHAPTER V

DIAGRAMS OF ALIGNMENT WITH TWO OR MORE INDICES

19. Diagrams of Double Alignment.—Sometimes a formula or an equation may be given the form

$$f_{12} = f_{34} \tag{38}$$

and may be replaced by a pair of equivalent equations

$$f_{12} = h \qquad {}_{34} = h$$

where h is an auxiliary variable. Assume now that each of these equations can be represented by a diagram of alignment. By determining the value of h from one diagram the value of either one of the remaining variables, say z_3 or z_4, could be found from the second diagram. If, however, both equations can be represented with the same scale for h, a single

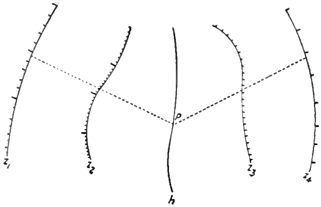


FIG. 63*a*.

figure with four z scales and one h scale would constitute a complete diagram for the original equation.

Such a diagram is called a *diagram of double alignment* or a *diagram of double collineation*. The scale for h is called the *hinge* or *pivot scale* and need not be graduated unless this is desirable for convenience in locating the temporary point about which the index is turned for its second position. The type of diagram and the way to the solution is shown in the schematic figure, Fig. 63*a*.

The diagram is often more conveniently arranged when the part including the z_1 and z_2 scales is superimposed in the other part of the figure, but in many cases where the scales are on parallel supports greater accuracy and ease of use will result when the pivot scale is chosen between the scales for each part of the diagram and the indices are placed in the form of a letter X as shown in Fig. 69, page 8.

Some thought should be given to the way in which the variables are grouped on either side of the equality sign in Equation (38) so that those variables which are perhaps more closely related or those which have about the same range of numbers may be used in the same half of the diagram. It is usually necessary to use different schemes of scale factors for each auxiliary equation and the only restriction on the equations is that they be of type (8) of Chapter III. It is always required, however, that the hinge scale have the same defining equations in every respect in order that the same value of h shall be determined in both diagrams by corresponding values of the variables.

Example 41.—The formula of Example 33 for the mean hydraulic radius of trapezoidal sections of canals will be arranged as an example of the diagram of double collineation. The formula is

$$R = H\,\frac{1 + K\cot\phi}{1 + 2K\operatorname{cosec}\phi} \qquad K = \frac{H}{b}$$

where R is the mean hydraulic radius, H the depth of water, b the breadth of canal bottom, and ϕ the angle of slope with the horizontal. The formula should first be rewritten in the form of Equation 38.

$$\frac{R}{H} = h = \frac{\sin\phi + K\cos\phi}{\sin\phi + 2K}$$

and then in the reduced determinant equation forms

$$\begin{vmatrix} 0 & h & 1 \\ -\tfrac{1}{2}\sin\phi & \tfrac{1}{2}\cos\phi & 1 \\ K & 1 & 1 \end{vmatrix} = 0 = \begin{vmatrix} 0 & h & 1 \\ 1 & -R & 1 \\ \dfrac{1}{1+H} & 0 & 1 \end{vmatrix}$$

Since K will never be greater than unity it is evident that the unit of the drawing for the first determinant must be fairly large and it may be chosen as 4 inches. The scale for the angle ϕ will then be a circle with a radius of one-half unit or two inches, which is ample, as the angles need not be measured closer than degrees. No scale factors are needed.

In the second equation it is seen that the R scale may be inconveniently long with the unit of the drawing as 4 inches and yet since the value of the H function is never greater than unity the horizontal scale must not be contracted. What is needed then is to extend the horizontal scale and contract the vertical scale and at the same time leave the h scale unchanged.

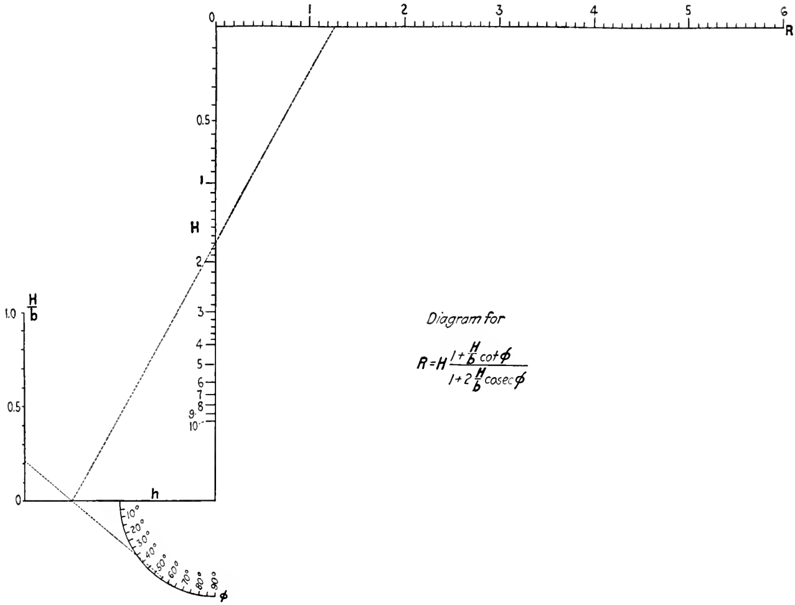


Fig. 64.

The result is accomplished with a little study by first writing the second equation in the form

$$\begin{vmatrix} 1 & -R & 1 \\ 0 & h & 1 \\ \frac{1}{1+H} & 0 & 1 \end{vmatrix} = 0$$

This equation is of the type (17) of Chapter III and Equations (21) are applicable to the defining equations. Sufficient contraction will result if μ_1 is taken $\frac{1}{2}$, then μ_2 must be unity in order to preserve the h scale intact. The scale for H may be extended and at the same time the position of the R scale improved by choosing $\delta = 2.5$. There results for the second set of defining equations

$$x = 2.5 \qquad y = -\frac{R}{2}$$
$$x = 0 \qquad y = h$$
$$x = \frac{5}{2+H} \qquad y = 0$$

The completed diagram is shown in Fig. 64 which has been rotated 90° to improve its position on the sheet. The limiting values of the variables chosen here are general and the graduations could be greatly refined for special work, for small drainage ditches for example.

Example 42.—Unwin's formula for the flow of steam in pipes

$$W = 87.5\sqrt{\frac{Dpd^5}{L\left[1+\frac{3.6}{d}\right]}}$$

where W = number of pounds flowing per minute
D = density in pounds per cubic foot
p = loss of pressure due to friction, in pounds per square inch

d = nominal inside diameter of pipe in inches
L = length of pipe in feet,

may be written

$$\frac{W}{87.5\sqrt{\frac{p}{L}}} = h = \sqrt{D}\sqrt{\frac{d^5}{1+\frac{3.6}{d}}}$$

If L is taken as 100 feet and p expressed as the loss in pressure per 100 feet of length, the formula is similar to Equation (38)

$$\frac{W}{8.75\sqrt{p}} = h = \sqrt{D}\sqrt{\frac{d^5}{1+\frac{3.6}{d}}}$$

or $\quad \log W - \log 8.75 - \frac{1}{2}\log p = \log h$

$$\tfrac{1}{2}\log D + \tfrac{1}{2}\log\left[\frac{d^5}{1+\frac{3.6}{d}}\right] = \log h$$

Both of these auxiliary equations are similar in form to Equation (10) and yield three parallel straight scales. The diagram is shown in Fig. 65 with the defining equations

$$\begin{array}{ll} x = -\delta_1 & y = \mu_1 \log W \\ x = \delta_2 & y = -\mu_2[\log 8.75 + \tfrac{1}{2}\log p] \\ x = 0 & y = \mu h \\ x = \delta_3 & y = \mu_3 \log\sqrt{\dfrac{d^5}{1+\frac{3.6}{d}}} \\ x = -\delta_4 & y = \mu_4 \log\sqrt{D} \end{array}$$

The proportion $\frac{\delta_1}{\delta_2} = \frac{\mu_1}{\mu_2} = \frac{1}{4} = \frac{\delta_3}{\delta_4} = \frac{\mu_3}{\mu_4}$ was used to give a symmetrical diagram with convenient scale lengths. Since D, the density, depends upon the steam pressure the corresponding pressures were plotted in place of the various densities. To the scale for W was added a scale for the approximate boiler horsepower. The indices show the setting to determine the flow in a 6-inch pipe carrying steam at 140 pounds gauge pressure allowing a drop of 3 pounds for each 100 feet.

Any equation or formula of the form

$$\frac{a^m}{p^r} = \frac{q^s}{b^n} \tag{39}$$

($m, n, r, s,$ = constants)

may be replaced by the equivalent system of equations

$$\frac{a^m}{p^r} = h \qquad \frac{q^s}{b^n} = h$$

and two corresponding first determinant equations are

$$\begin{vmatrix} 1 & h & 0 \\ 0 & -a^m & 1 \\ p^r & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & h & 0 \\ 0 & -q^s & 1 \\ b^n & 0 & 1 \end{vmatrix} = 0 \tag{40}$$

The choice of the reduced determinant forms of these equations may be made by first adding either the first or the second columns to the third to form a new third column in each determinant. The choice will of course be made with a view to the economy of calculation for the resulting scales on the X or Y axis. Equations (40) are of the type (14) of Chapter III.

The above equation (39) may be written

$$m \log a - r \log p = \log h = s \log q - n \log b$$

and the two auxiliary equations will be similar to Equation (10) and require simply four logarithmic scales on as many parallel straight lines. The determinant equations are

$$\begin{vmatrix} -1 & m\log a & 1 \\ 1 & -r\log p & 1 \\ 0 & \log h & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & s\log q & 1 \\ 1 & -n\log b & 1 \\ 0 & \log h & 1 \end{vmatrix} = 0$$

and the defining equations of Section 12 including δ and μ apply to each.

It is to be observed that no plotting on the h scale is necessary but the same value of h must determine the same point on the h scale so that the reduced determinant forms of the equations must both result in the same defining equations for the h scale even though that scale is not graduated. It must therefore be borne in mind that the choice of the scale factor for the h scale must be the same for both equations.

Example 43.—As an illustrative example of the above Equation (39) Chezy's formula for the flow of water in open channels may be studied. The formula is

$$V = c\sqrt{RS}$$

where V, R, and S have the designations of Example 37 and c is Chezy's coefficient. The form for the reduced determinant equations may be taken

$$\begin{vmatrix} 1 & h & 1 \\ 0 & -V & 1 \\ \dfrac{c}{c+1} & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & h & 1 \\ 0 & -R^{\frac{1}{2}} & 1 \\ \dfrac{S^{-\frac{1}{2}}}{1+S^{-\frac{1}{2}}} & 0 & 1 \end{vmatrix} = 0$$

It will be necessary to graduate the scales for R and V and for S and c on the same axes, and this will always be necessary for equations of the type (40). Since the c and S scales will not extend beyond unit distance from the origin it will be well to make δ as

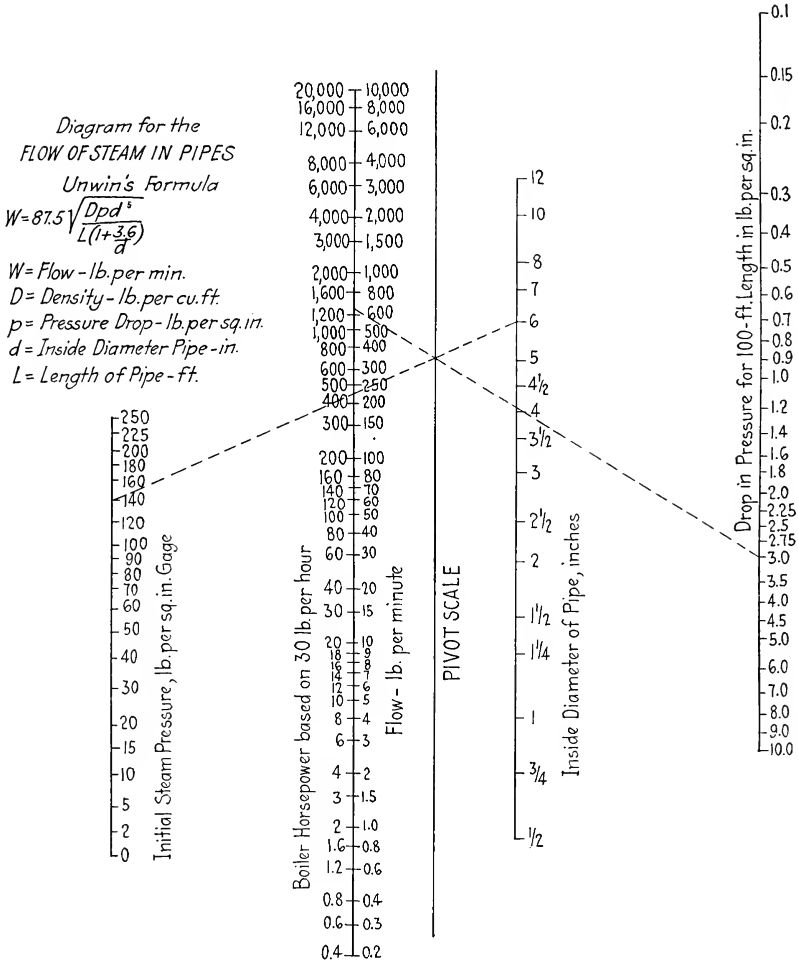


Diagram for the
FLOW OF STEAM IN PIPES
Unwin's Formula
$W = 87.5\sqrt{\frac{Dpd^5}{L(1+\frac{3.6}{d})}}$
W = Flow - lb. per min.
D = Density - lb. per cu. ft.
p = Pressure Drop - lb. per sq. in.
d = Inside Diameter Pipe - in.
L = Length of Pipe - ft.
Initial Steam Pressure, lb. per sq. in. Gage
250 225 200 180 160 140 120 100 90 80 70 60 50 40 30 20 15 10 5 2 0
Boiler Horsepower based on 30 lb. per hour
20,000 16,000 12,000 8,000 6,000 4,000 3,000 2,000 1,600 1,200 1,000 800 600 500 400 300 200 160 140 120 100 80 60 40 30 20 18 16 14 12 10 8 6 4 3 2 1.6 1.2 0.8 0.6 0.4
Flow - lb. per minute
10,000 8,000 6,000 4,000 3,000 2,000 1,500 1,000 800 600 500 400 300 250 200 150 100 80 70 60 50 40 30 20 15 10 9 8 7 6 5 4 3 2 1.5 1.0 0.8 0.6 0.4 0.3 0.2
PIVOT SCALE
Inside Diameter of Pipe, inches
12 10 8 7 6 5 4½ 4 3½ 3 2½ 2 1½ 1¼ 1 ¾ ½
Drop in Pressure for 100-ft. Length in lb. per sq. in.
0.1 0.15 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.2 1.4 1.6 1.8 2.0 2.25 2.5 2.75 3.0 3.5 4.0 4.5 5.0 6.0 7.0 8.0 9.0 10.0

FIG. 65.

large as the sheet will permit. Also since the h scale and the V and R scales are in opposite directions the X axis may well be chosen at an acute angle with the Y axis at the outset.

If the formula is written

$$\log V - \log c = \log h = \tfrac{1}{2} \log R + \tfrac{1}{2} \log S$$

the two determinant equations are

$$\begin{vmatrix} -1 & \log V & 1 \\ 1 & -\log c & 1 \\ 0 & \log h & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & \tfrac{1}{2} \log R & 1 \\ 1 & \tfrac{1}{2} \log S & 1 \\ 0 & \log h & 1 \end{vmatrix} = 0$$

and the general arrangement of the figure would be similar to Fig. 65 of Example 42. This latter form of similar formulas usually results in simpler and easier plotting.

Some formulas in four or more variables can best be handled by combinations of simple diagrams composed of straight line systems as described in Chapter II. The use of such diagrams is very common and obviously all that has been said regarding the use of a hinge scale h for combinations of collinear diagrams applies equally well in such cases. (See Problem 16 of Chapter III.) Often a simple diagram may be combined with a collinear diagram so that the scale on one of the axes of the simple diagram serves also as the h scale of the collinear diagram. When two simple diagrams are placed "back to back" or superimposed, of course no hinge scale is used.

20. Diagrams with Parallel or Perpendicular Indices.—Such diagrams consist of four scales

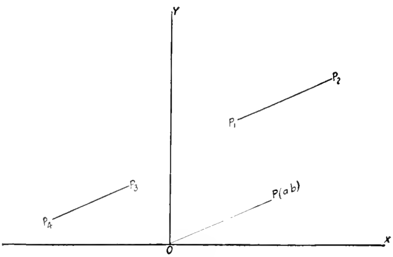


FIG. 66.

arranged in pairs corresponding to the four variables of an equation or formula. The scales are so disposed that a straight line drawn through a known point on a third scale parallel or perpendicular to a line joining two known points on two other scales will cut the fourth scale in the point inscribed with the value of the unknown variable. The geometric theory involved is as follows: The equation

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$$

expresses the equality of the slopes of the two lines P_1P_2 and OP respectively as shown in Fig. 66.

This equation above may be written in the determinant form

$$\begin{vmatrix} a & b & 0 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

and another equation

$$\begin{vmatrix} a & b & 0 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = 0$$

regarded as a simultaneous equation would then express the fact that the lines P_1P_2 and P_3P_4 were parallel. If a and b are eliminated from the above two determinant equations there results

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

Consider now an equation in four variables which has the form

$$\frac{g_2 - g_1}{f_2 - f_1} = \frac{g_4 - g_3}{f_4 - f_3} \tag{41}$$

This equation may be regarded as the result of eliminating h from the two determinant equations

$$\begin{vmatrix} 1 & h & 0 \\ f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & h & 0 \\ f_3 & g_3 & 1 \\ f_4 & g_4 & 1 \end{vmatrix} = 0 \tag{42}$$

Consequently the straight line index of a diagram with the defining equations

$$x = f_1 \qquad y = g_1$$
$$x = f_2 \qquad y = g_2$$

will be parallel to the index of a diagram plotted with the same coordinate axes and with the defining equations

$$x = f_3 \qquad y = g_3$$
$$x = f_4 \qquad y = g_4$$

since both indices will have the same slope whenever Equation (41) or the equivalent system (42) is satisfied by a set of values of the variables $z_1 \ldots z_4$.

Example 44.—Lamé's formula for thick cylinders may be arranged to afford an example of the use of diagrams with parallel indices. As usually given the formula is

$$D = d\sqrt{\frac{S + P}{S - P}}$$

where the letters have the meanings given on the dia-

gram for the formula shown in Fig. 67.

The formula may be written

$$\frac{S+P}{S-P} = \frac{D^2 - 0}{0 + d^2}$$

and may be regarded as the result of eliminating h from the two simultaneous equations

$$\begin{vmatrix} 1 & h & 0 \\ P & -P & 1 \\ S & S & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & h & 0 \\ -d^2 & 0 & 1 \\ 0 & D^2 & 1 \end{vmatrix} = 0$$

A set of defining equations for both these equations may at once be written without reference to the first rows involving the auxiliary variable h but usually scale factors will be needed and since the parallelism of all lines must be preserved the scale factors for defining equations of the second set must be proportional to those of the first set thus

$$\begin{array}{llll} x = \delta_1 P & y = -\mu_1 P & x = -\delta_2 d^2 & y = 0 \\ x = \delta_1 S & y = \mu_1 S & x = 0 & y = \mu_2 D^2 \end{array}$$

where

$$\frac{\delta_1}{\mu_1} = \frac{\delta_2}{\mu_2}$$

The method used in this example is general. The underlying principle is the use of a projective transformation that preserves parallelism. (See Appendix B.)

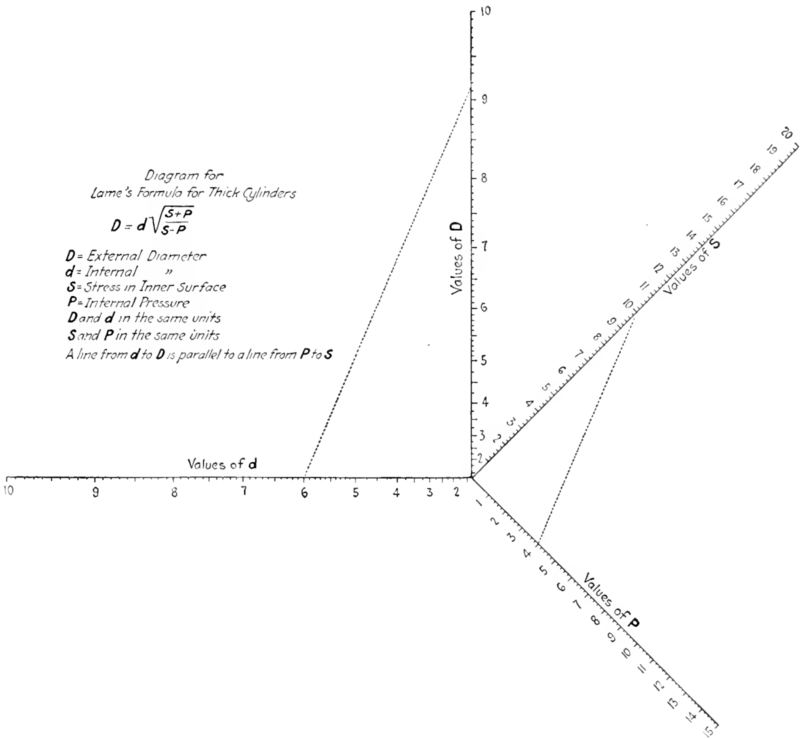


Fig. 67.

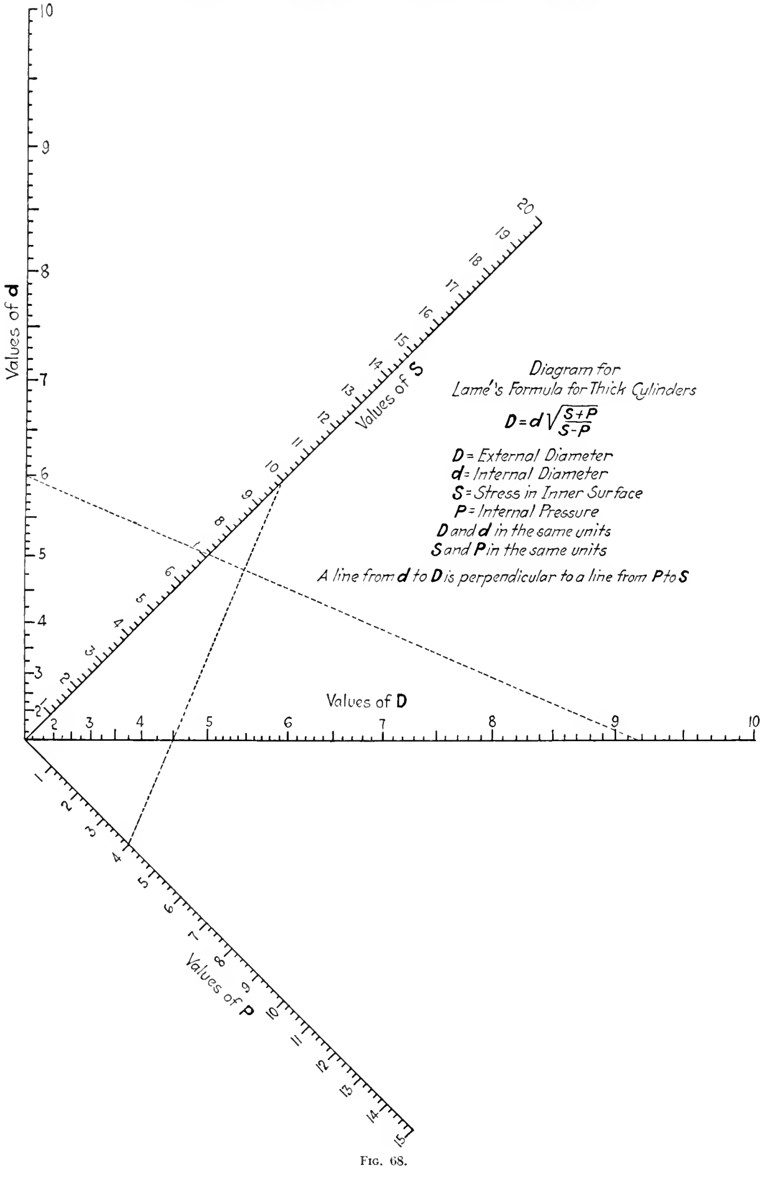

Values of d
Values of S
Values of D
Values of P
Diagram for
Lamé's Formula for Thick Cylinders
$D = d\sqrt{\frac{S+P}{S-P}}$
D = External Diameter
d = Internal Diameter
S = Stress in Inner Surface
P = Internal Pressure
D and d in the same units
S and P in the same units
A line from d to D is perpendicular to a line from P to S

Fig. 68.

In the present example the scales are all straight and readily plotted. The same units must be used for S and P such as tons or thousands of pounds: also in using the D and d scales the same units must be employed, as inches or centimeters. The indices are shown set for $P = 4{,}000$, $S = 10{,}000$, $d = 6$ required D. Reading of the diagram may sometimes be made more convenient by providing a transparent piece of celluloid on which parallel lines are scratched.

It is now quite evident that Equation (39) of Article 18 may be represented also by a diagram with parallel indices. In fact Equations (40) constitute the necessary reduced determinant equations. This Equation (39) serves also to show that where the scales of Equation (42) reduce to straight scales supported respectively in pairs on the same straight lines, the necessary theory of the parallel alignment of the indices is merely the geometry of similar triangles. In case the supports of the scales are parallel the segments intercepted by the two indices are proportional. Those who are familiar with the use of homogeneous coordinates in geometry will recognize that the presence of zero in place of unity in the third element position of the determinants of Equations (42) merely indicates that the resulting diagram with parallel indices is a special case of the diagrams of double collineation in which the hinge scale has been removed to infinity.

It is possible to give equations of the form (42), which includes Equation (40) as a special case, another simple diagrammatic representation. In this representation the key to the solution is by perpendicular indices instead of by parallel indices and it has some advantage because of the fact that two perpendicular lines scratched on a piece of transparent celluloid will serve as the two perpendicular indices and both pairs of scales may be read at one setting. Bearing in mind that h in both determinants of Equation (42) represents the variable slope of the two indices it is only necessary to replace unity in the first element position of the second (or first) determinant by $-h$, and h in the second element position of the first row by unity in order that the slopes of the two indices shall be no longer equal but one the negative reciprocal of the other whenever they are to determine corresponding values of the four variables $z_1 \ . \ . \ . \ z_4$. The corresponding change in Equation (41) requires that equation to be written

$$\frac{g_2 - g_1}{f_2 - f_1} = -\frac{f_4 - {}_{\cdot}{}_3}{g_4 - g_3} \tag{41a}$$

but the defining equations are selected from this equation exactly as they were for Equation (41). In other words the original Equation (41) may be represented by a diagram with perpendicular indices by writing first the two determinant equations

$$\begin{vmatrix} 1 & h & 0 \\ f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & -\frac{1}{h} & 0 \\ g_3 & -f_3 & 1 \\ g_4 & -f_4 & 1 \end{vmatrix} = 0 \tag{42a}$$

as a check and constructing the diagram from the defining equations

$$\begin{array}{llll} x = f_1 & y = g_1 & x = g_3 & y = -f_3 \\ x = f_2 & y = g_2 & x = g_4 & y = -f_4 \end{array}$$

Example 45.—From the formula of the preceding example another set of defining equations may be written which will yield a diagram with perpendicular indices as follows

$$\begin{array}{llll} x = \delta_1 P & y = -\mu_1 P & x = \delta_2 D^2 & y = 0 \\ x = \delta_1 S & y = \quad \mu_1 S & x = 0 & y = \mu_2 d^2 \end{array}$$

and the diagram is shown in Fig. 68.

21. Diagrams for the Equation $f_1 + f_2 \ . \ . \ . + f_n = 0$. —Sometimes the Equation (38) in Article 19 has the simple form

$$f_1 + f_2 = f_3 + f_4 \tag{43}$$

and a diagram of double collineation with parallel straight scales may be constructed by using as before the auxiliary variable h and the two equivalent equations

$$f_1 + f_2 - h = 0 \qquad\qquad f_3 + f_4 - h = 0$$

The corresponding determining equations with suitable scale factors will then be (Article 12, Chapter III)

$$\begin{array}{llll} x = -\delta_1 & y = \mu_1 f_1 & x = -\delta_3 & y = \mu_3 f_3 \\ x = \quad \delta_2 & y = \mu_2 f_2 & x = \quad \delta_4 & y = \mu_4 f_4 \\ x = \quad 0 & y = \dfrac{\mu_1\mu_2 h}{\mu_1 + \mu_2} & x = \quad 0 & y = \dfrac{\mu_3\mu_4 h}{\mu_3 + \mu_4} \end{array}$$

where it is necessary in addition to the conditions imposed upon the constants μ and δ in Article 12 to require also that

$$\frac{\mu_1\mu_2}{\mu_1 + \mu_2} = \frac{\mu_3\mu_4}{\mu_3 + \mu_4} \tag{44}$$

in order that the same value of the auxiliary variable h shall always determine the same point on the hinge scale. The scheme of the resulting diagram is shown in Fig. 69 and Example 42 illustrated its application.

If it is found more convenient the determinant equations used for the Equation (43) may be written

$$\begin{vmatrix} 1 & \frac{h}{2} & 0 \\ -1 & -f_1 & 1 \\ 1 & f_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & \frac{h}{2} & 0 \\ -1 & -f_3 & 1 \\ 1 & f_4 & 1 \end{vmatrix} = 0$$

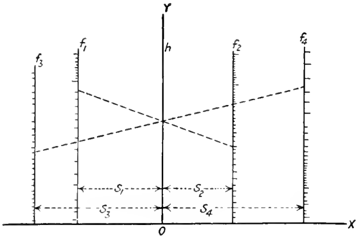


FIG. 69.

and a diagram with parallel indices designed. The defining equations with scale factors are

$$x = -\delta_1 \quad y = -\mu_1 f_1 \quad x = -\delta_2 \quad y = -\mu_2 f_3$$
$$x = \delta_1 \quad y = \mu_1 f_2 \quad x = \delta_2 \quad y = \mu_2 f_4$$

where

$$\frac{\delta_1}{\mu_1} = \frac{\delta_2}{\mu_2}$$

If $\delta_1 = \delta_2$ then of necessity $\mu_1 = \mu_2$, and the supports of the scales coincide in pairs and the resulting diagram is shown in Fig. 70.

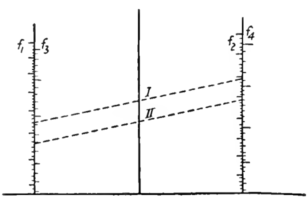


FIG. 70.

New types of diagrams may now be constructed for the equation (7) of n variables

$$f_1 + f_2 + f_3 + \;.\;.\;.\; + f_n = 0 \qquad (7)$$

Introduce the auxiliary variables $h_1, h_2, \;.\;.\;.\;$ etc., and write the equivalent system of equations

$$f_1 + f_2 - h_1 = 0$$
$$h_1 + f_3 - h_2 = 0$$
$$h_2 + f_4 - h_3 = 0$$
$$.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$
$$h_{n-3} + f_{n-1} + f_n = 0$$

In each of these equations except the first one and the last one two auxiliary variables h enter and one value of h must always have a support for each application of the index in the diagram. For example h_2 may be determined from the first two equations written in the form

$$\begin{vmatrix} 1 & \frac{h_1}{2} & 0 \\ -1 & -f_1 & 1 \\ 1 & f_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & \frac{h_1}{2} & 0 \\ -1 & f_3 & 1 \\ 1 & h_2 & 1 \end{vmatrix} = 0$$

and represented by a corresponding diagram with parallel indices in which no support appears for h_1 but in which one does appear for h_2. Figure 71 shows the scheme of such a diagram.

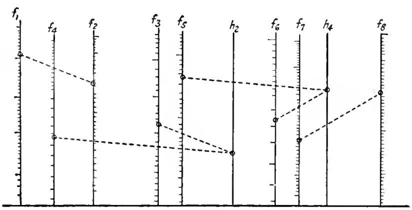


FIG. 71.

It is not necessary of course to use the principle of parallel indices to construct the diagram for Equation (7) as hinge scales can be used throughout, but is frequently convenient to do so. The spacing of the scales and the use of the scale factors are controlled at each step by principles already laid down in this chapter and in Chapter III.

Example 46.—The formula

$$T = 0.2618 \frac{LFD}{S}$$

for determining the actual time for turning a piece of work in a lathe is shown in Fig. 72 and the symbols of the formula are described on the figure. If written

$$\log T + \log S - \log 0.2618 - \log F - \log L - \log D = 0$$

the formula is in a form similar to Equation (7). If P, the product LD is introduced as an auxiliary variable the equations are

$$\log T + \log S = \log h = \log F + \log P + \log 0.2618$$
$$\log P = \log L + \log D$$

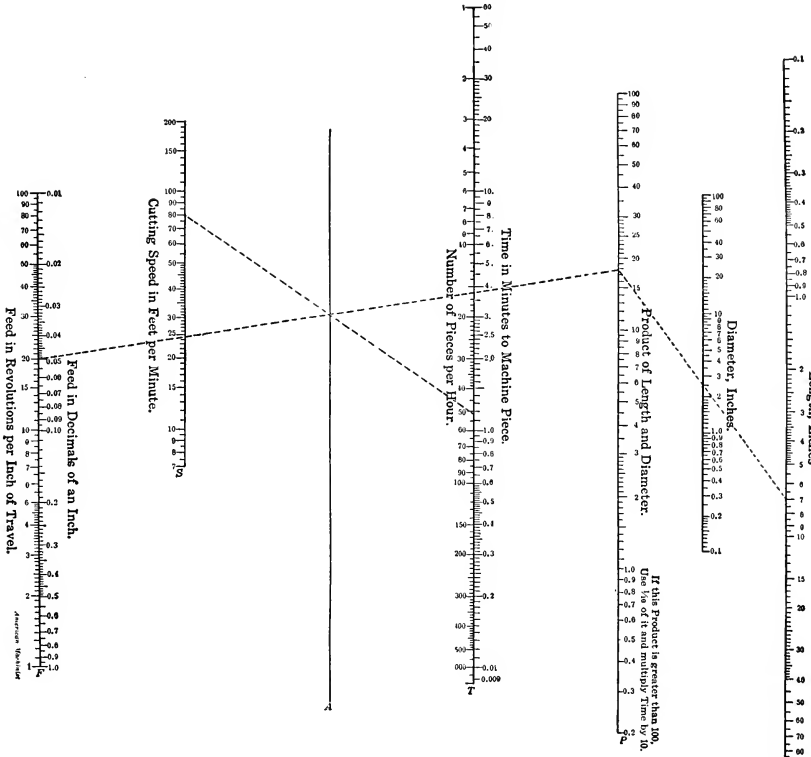


FIG. 72.—Diagram for $T = 0.2618 \frac{LFD}{S}$.

The first equation yields the following defining equations

$$x = -2\delta \qquad y = \mu \log F$$
$$x = 2\delta \qquad y = \mu \log P + \mu \log 0.2618$$
$$x = 0 \qquad y = \frac{\mu}{2} \log h$$
$$x = -\delta \qquad y = \mu \log S$$
$$x = \delta \qquad y = \mu \log T$$

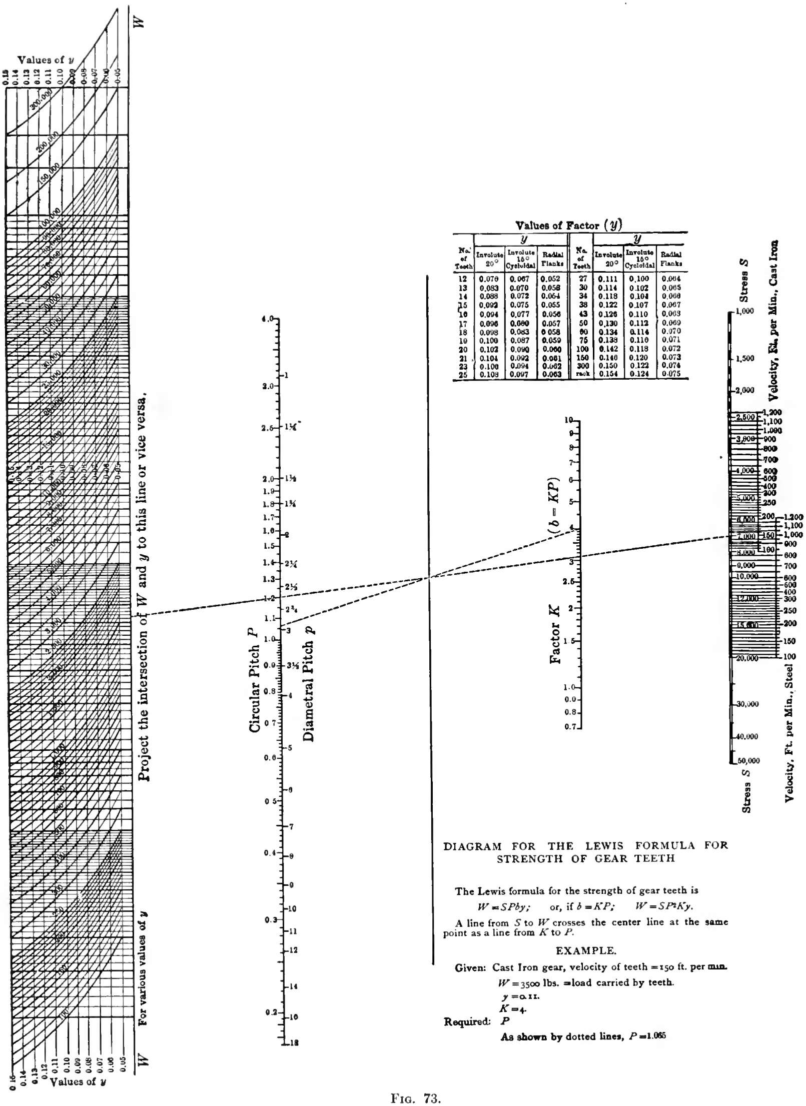
The equation $P = LD$ if plotted from its form as given above, would require that the scale for P be located between the scales for D and L. Rearranging it in the form

$$\log P - \log L = \log D$$

results in placing the D scale between the other two scales thus permitting the auxiliary diagram for $DL = P$ to be joined conveniently at the side of the diagram for the first equation instead of superimposing it upon the latter.

The defining equations for the second equation referred to an origin on the D scale are

$$x = -\delta_1 \qquad y = \mu \log P$$
$$x = 0 \qquad y = \frac{\mu}{2} \log D$$
$$x = \delta_1 \qquad y = \mu \log L$$



Values of Factor (y)

No. of Teeth	y Involute 20°	y Involute 15° Cycloidal	y Radial Flanks	No. of Teeth	y Involute 20°	y Involute 15° Cycloidal	y Radial Flanks
12	0.078	0.067	0.052	27	0.111	0.100	0.064
13	0.083	0.070	0.053	30	0.114	0.102	0.065
14	0.088	0.072	0.054	34	0.118	0.104	0.066
15	0.092	0.075	0.055	38	0.122	0.107	0.067
16	0.094	0.077	0.056	43	0.126	0.110	0.068
17	0.096	0.080	0.057	50	0.130	0.112	0.069
18	0.098	0.083	0.058	60	0.134	0.114	0.070
19	0.100	0.087	0.059	75	0.138	0.116	0.071
20	0.102	0.090	0.060	100	0.142	0.118	0.072
21	0.104	0.092	0.061	150	0.146	0.120	0.073
23	0.106	0.094	0.062	300	0.150	0.122	0.074
25	0.108	0.097	0.063	rack	0.154	0.124	0.075

DIAGRAM FOR THE LEWIS FORMULA FOR STRENGTH OF GEAR TEETH

The Lewis formula for the strength of gear teeth is

$W = SPby$; or, if $b = KP$; $W = SP^2Ky$.

A line from S to W crosses the center line at the same point as a line from K to P.

EXAMPLE.

Given: Cast Iron gear, velocity of teeth = 150 ft. per min.

$W = 3500$ lbs. = load carried by teeth.

$y = 0.11$.

$K = 4$.

Required: P

As shown by dotted lines, $P = 1.065$

Fig. 73.

Example 47.—In the Lewis formula for the strength of gear teeth

$$W = SPb(y)$$

it is often desired to solve for P if b is taken as KP, where

W = load in pounds carried by the teeth
S = stress in pounds per square inch, chosen with reference to the velocity and material of the teeth
P = circular pitch in inches
b = width of face in inches
K = a constant, usually 2 to 6
(y) = a factor depending upon the number and shape of the teeth.

In logarithmic form the formula becomes

$$\log W - \log S - 2 \log P - \log K - \log (y) = 0$$

but if the auxiliary variable h_1 is chosen so that

$$h_1 = \frac{W}{(y)}$$

it becomes

$$\log h_1 - \log S = \log h = 2 \log P + \log K$$

Figure 73 was then plotted from the defining equations

$$\begin{array}{ll} x = -2\delta & y = \mu \log h_1 = \mu \log \frac{W}{(y)} \\ x = 2\delta & y = -\mu \log S \\ x = 0 & y = \frac{\mu}{2} \log h \\ x = -\delta & y = 2\mu \log P \\ x = \delta & y = \mu \log K \end{array}$$

The first two equations define a binary scale on the line $x = -2\delta$. The curve net drawn for this scale may consist of the lines

$$x = -(y),\ y = \mu \log \left[\frac{W}{-x}\right]$$

plotted with the line $x = -2\delta$ as a new Y axis. (See Article 16, Chapter IV.)

From a table given by Mr. Lewis the values of the velocity were added in proper correspondence with the scale for S. Since the product of the diametral pitch and the circular pitch is always π a scale for the diametral pitch was added to the P scale.

The equation $h_1 = \frac{W}{(y)}$ could naturally be written in the logarithmic form and two auxiliary (collinear) logarithmic scales for W and (y) added to the present figure just as was done for L and D in Fig. 72 for the preceding example. The range of numbers for (y), however, is very small and it was found more convenient to use the system of curves and establish a binary scale on the line $x = -2\delta$ as shown in the figure.

Problem 1.—Discuss the equation of Example 47 as an equation of type (7) and construct the diagram resulting when the binary scale is replaced by the required parallel scales for W and (y).

Problem 2.—Reduce Bazin's equation for the velocity of water to type (7) and construct a corresponding diagram.

Problem 3.—Construct a practical diagram for the formula of Example 12 of Chapter II. Write useful values of c and use four parallel lines.

Problem 4.—Gordon's formula for columns is

$$W = \frac{5a}{1 + \frac{l^2}{600d^2}}$$

where W = safe unit load, 2,725 to 14,450
a = coefficient, 2,000 to 3,200
l = unsupported length in inches
d = least dimension in inches
$\frac{l}{d}$ = 8 to 40.

If the determinant equation for l and d is

$$\begin{vmatrix} 1 & h & 0 \\ \frac{1 - 600d^2}{600\, d^2} & 0 & 1 \\ -\frac{1 + l^2}{l^2} & \frac{-1}{l^2} & 1 \end{vmatrix} = 0$$

find the corresponding determinant equations for a and W and design a diagram of parallel alignment with suitable scale factors for practical use in steel design.

Problem 5.—In Section 15 of Chapter III were described diagrams of alignment with a fixed point; show that Equation (41) can be represented by a diagram with the fixed point $x = 1$, $y = 0$ and two binary scales on the Y axis.

Problem 6.—A reduction formula used in automobile radiation tests is

$$H_{200-70} = \frac{62.4A}{1 + \frac{0.24A}{W} + \frac{0.48AD_1}{H_1}}$$

where A = (6,000 to 15,000) air, pounds per hour.
D_1 = (80° to 115°F) mean temperature difference
H_1 = (40,000 to 90,000) heat transfer observed, B.t.u.
W = (1,000 to 3,500) water, pounds per hour.

Show how to design a diagram of double alignment with parallel scales for this formula and with the quantities grouped in the pairs H, A, and W, K where $K = \frac{A}{D_1}$.

CHAPTER VI

ALIGNMENT DIAGRAMS WITH ADJUSTMENT

Introduction.—There is introduced in this chapter a new class of diagrams based upon fundamental principles already developed. These diagrams enlarge the number of types of equations to which the principle of collineation is immediately applicable and furnish also a general alternative method especially for those equations which cannot readily be identified with preceding types. It will moreover be found that the types of equations already treated may be regarded as special cases of the more general types now to be discussed.

22. Equations in Three Variables.—It was shown in Chapter III that an equation in three variables

$$f_{123} = 0$$

may be represented by a collinear diagram with three scales when and only when it can be written in the reduced determinant form

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \qquad (8)$$

Since there is no immediate and satisfactory test for this desired property of an equation or formula it is usually necessary to resort to the principle of comparison with certain type forms and to the tentative rule of Article 14. It is therefore desirable to free the determinant form from restrictions as far as possible and at the same time to preserve the principle of collineation, or the use of the straight line index in designing the diagram.

The fundamental property of Equation (8) is the segregation of the variables f_1, f_2, f_3, into *rows* of the determinant.

Consider now the determinant equation of the type

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{23} & g_{23} & 1 \\ f_{31} & g_{31} & 1 \end{vmatrix} = 0 \qquad (45)$$

in which the elements of each row are allowed to contain at most two variables, which variables may appear in more than one row.

Then the equations analogous to the previously defined and much used defining equations will be of the type

$$\begin{aligned} x &= f_{12} \qquad & y &= g_{12} \\ x &= f_{23} \qquad & y &= g_{23} \\ x &= f_{31} \qquad & y &= g_{31} \end{aligned} \qquad (46)$$

and each pair of equations will determine a curve net, except in the case explained below in Article 23. The three curve nets are shown schematically in Fig. 74 and it is seen that there appear two families of curves for each variable z.

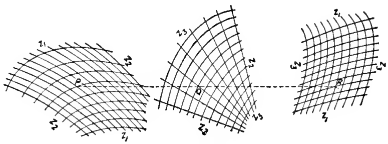


FIG. 74.

Call any set of three values of z which simultaneously satisfy Equation (45) *corresponding* values of z. Such values of z necessarily determine values of the functions f and g and hence by Equations (46) there result three pairs of coordinates x and y which must satisfy Equation (45) when substituted for f and g. But Equation (45) would then express the geometric fact that three points in the three respective curve nets are collinear.

In general, however, it would not be true that any three points taken at random in the three plotted curve nets and on the same straight line would yield corresponding values of z attached to the curves intersecting in pairs at the respective points. It is here that the present theory departs from the theory previously developed. What happens in general is that there appear six values of z consisting of three pairs of dissimilar values.

When *corresponding* values of z are used to select three points in the three curve nets it is seen at once that the same value of z is used to select a curve from two different nets. If now one variable z_2 is unknown

it is evident that the index must be rotated about the point always determined in one net by the known values $z_1 z_3$ until the *same* value of the unknown z_2 appears in the two remaining nets at the points of intersection of the index with the known curves in each net. In the figure the line PR is rotated about R until the points of intersection P and Q determine the same value of the variable z_2 when it is assumed that z_1 and z_3 are known.

This then is the principle of collinear diagrams with adjustment. There are many special cases and in not a few no adjustment of the index is required because the unknown variable appears but once. The method is of great practical advantage especially if a given equation is not adapted to the preceding treatment.

23. Special Forms of Equations.—Equation (45) is a general form and is less frequent than the simpler special cases. For example f_{31} and g_{31} may often reduce to f_3 and g_3 respectively by skillful choice of the elements of the determinant. The corresponding determinant equation is then

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{23} & g_{23} & 1 \\ f_3 & g_3 & 1 \end{vmatrix} = 0 \qquad (47)$$

A simpler form of diagram results from this equation. Without scale factors the defining equations are

$$x = f_{12} \qquad y = g_{12}$$
$$x = f_{23} \qquad y = g_{23}$$
$$x = f_3 \qquad y = g_3$$

The first pair of equations lead to the curve net,

$$F_1(xy) = z_1 \qquad F_2(xy) = z_2$$

Similarly from the second pair is obtained the curve net

$$G_2(xy) = z_2 \qquad G_3(xy) = z_3$$

and the third pair of equations determine a curved scale for z with the support

$$S(xy) = 0$$

Still more simple is the equation

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_{23} & g_{23} & 1 \end{vmatrix} = 0 \qquad (48)$$

In the resulting diagram there will be two scales and a curve net defined as follows

$$x = f_1 \qquad y = g_1$$
$$x = f_2 \qquad y = g_2$$
$$x = f_{23} \qquad y = g_{23}$$

Frequently a redundancy of variables in an equation may be reduced by the introduction of a parameter which is a simple function of two or more of the variables whose values are always given, as was done in Chapter III in the case of Example 33 for the mean hydraulic radius of trapezoids. This device will be of advantage in several of the examples which follow.

Example 48.—As a first illustrative example of the Equation (48) the quadratic equation

$$z^2 + a_1 z + a_2 = 0$$

may be written in the reduced determinant form

$$\begin{vmatrix} -a_1 & 0 & 1 \\ 0 & z & 1 \\ z & k & 1 \end{vmatrix} = 0$$

where $k = -\dfrac{a_2}{a_1}$ is a parameter. The three variables are a_1, z, and k. The defining equations are

$$x = -a_1 \qquad y = 0$$
$$x = 0 \qquad y = z$$
$$x = z \qquad y = k$$

To solve a quadratic equation by this diagram the figure is entered on the X axis with the value of a_1 at P and the index is then turned about this point until the value of z read on the Y axis is the same as the value read on the vertical line intersecting the index where it is crossed by $y = -\dfrac{a_2}{a_1}$. In Fig. 75 the index is set for the two roots of the equation

$$z^2 - 6.2z - 18.6 = 0$$

Another simple case of equation (45) is

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_3 & g_3 & 1 \\ f_3' & g_3' & 1 \end{vmatrix} = 0 \qquad (49)$$

Example 49.—As an illustration of this form of an equation, the equation

$$z_1 z_2 - z_3 + \sqrt{1 + z_2^2} \cdot \sqrt{1 + z_1^2} = 0$$

which can not by algebraic transformation be given the form of Equation (8), may now be considered. It may at once be written in the first determinant form

$$\begin{vmatrix} 1 & 0 & z_1 \\ 0 & 1 & \sqrt{1 + z_1^2} \\ z_2 & \sqrt{1 + z_2^2} & z_3 \end{vmatrix} = 0$$

from which by adding columns one and two for a new second column and then interchanging columns two and three and dividing by the elements of column three there results the reduced determinant equation

$$\begin{vmatrix} 1 & z_1 & 1 \\ 0 & \sqrt{1 + z_1^2} & 1 \\ \dfrac{z_2}{z_2 + \sqrt{1 + z_2^2}} & \dfrac{z_3}{z_2 + \sqrt{1 + z_2^2}} & 1 \end{vmatrix} = 0$$

For a good arrangement of the drawing the vertical

unit may be taken one-tenth of the horizontal unit and there follow the defining equations

$$x = 1 \qquad y = \frac{z_1}{10}$$

$$x = 0 \qquad y = \frac{\sqrt{1 + z_1^2}}{10}$$

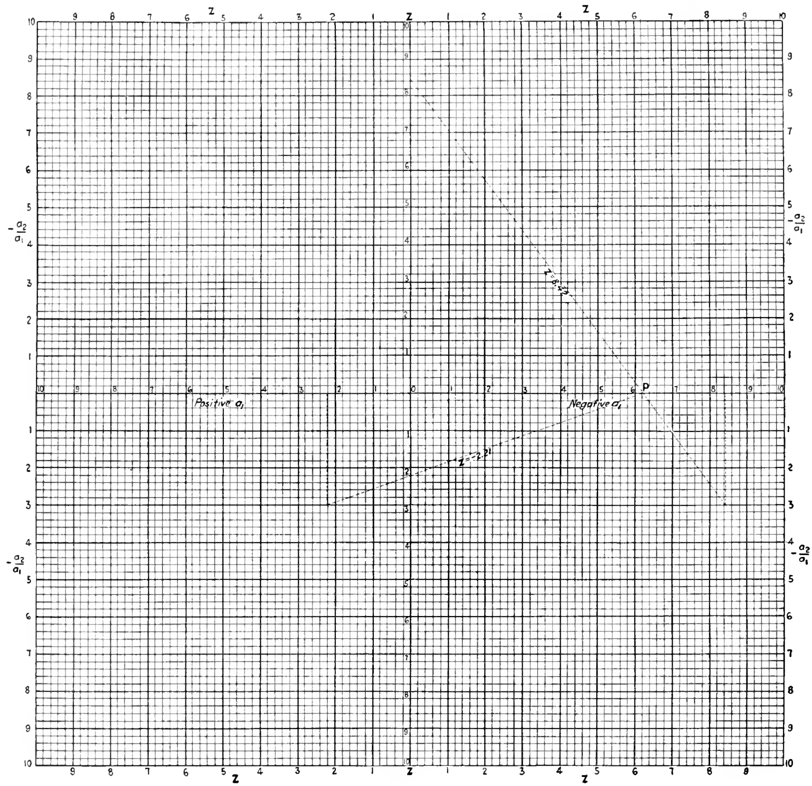


FIG. 75.

$$x = \frac{z_2}{z_2 + \sqrt{1 + z_2^2}} \qquad y = \frac{z_3}{10\,(z_2 + \sqrt{1 + z_2^2})}$$

The two scales for z_1 are easily drawn. For the curve net for z_2, z_3, since the right side of the first equation involves only z_2, there result straight lines parallel to the Y axis for the curves of that variable. The elimination of the variable z_2 between the last two equations leads to the equation of a family of parabolas

$$y^2 = \frac{z_3^2}{100}(1 - 2x)$$

for segregating the values of z_3. The finished diagram is shown in Fig. 76.

In constructing diagrams of adjustment it is desirable if possible to avoid binary scales as the use of the diagram is complicated by their presence.

Example 50.—The quadratic equation

$$z^2 + a_1 z + a_2 = 0$$

may be written in the form

$$\begin{vmatrix} -1 & a_1 z & 1 \\ 0 & \dfrac{-z^2}{2} & 1 \\ 1 & a_2 & 1 \end{vmatrix} = 0$$

which is another special case of Equation (45). With a horizontal unit twice the vertical the defining equations become

$$x = -2 \qquad y = a_1 z$$
$$x = 0 \qquad y = \frac{-z^2}{2}$$
$$x = 2 \qquad y = a_2$$

A binary scale is required on the line $x = -2$ and the variables a_1 and z may be segregated by setting

$$y = zx \qquad x = a_1$$

the resulting diagram is shown in Fig. 77.

The roots are determined by turning the index about the point P on the a_2 scale until the same values of z appear at Q and R and the operation is somewhat difficult to manage. In the figure the index is set twice for the roots of the equation

$$z^2 - 0.8z - 6.6 = 0$$

It will be seen that whenever any variable z appears in but one row of the determinant equation of the form (45), no adjustment of the index is necessary in determining its value from the corresponding diagram.

It is to be noticed that the successive elimination of two variables z_1 and z_2 from two equations

$$x = f_{12}, \; y = g_{12},$$

will fail if the two functions f_{12} and g_{12} are not

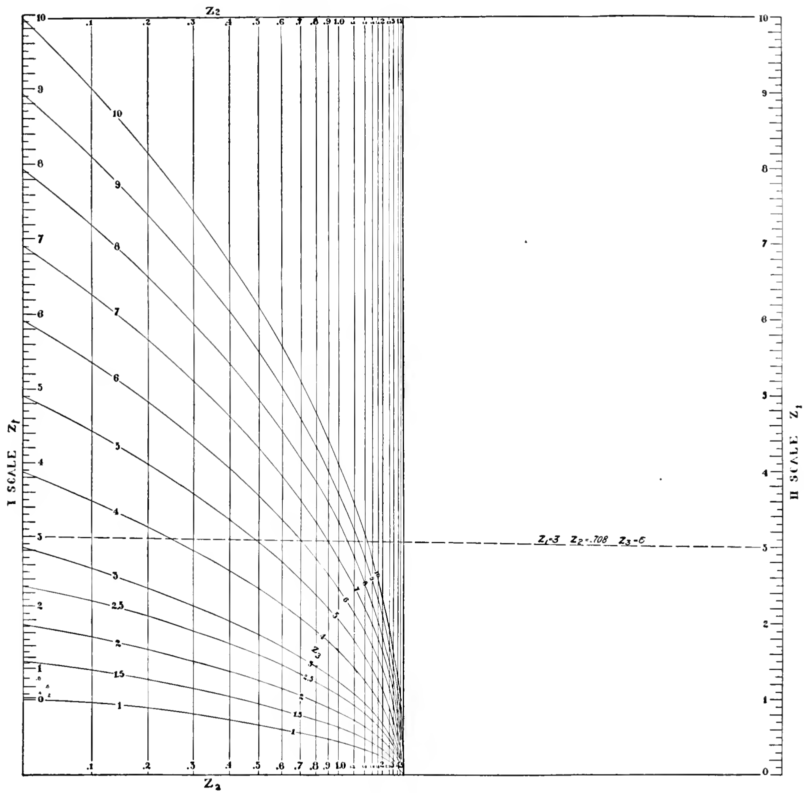


Fig. 76.

independent functions; that is to say in case one is a function of the other. Sometimes this condition plainly arises because both functions are functions of the same combination of the two variables z.

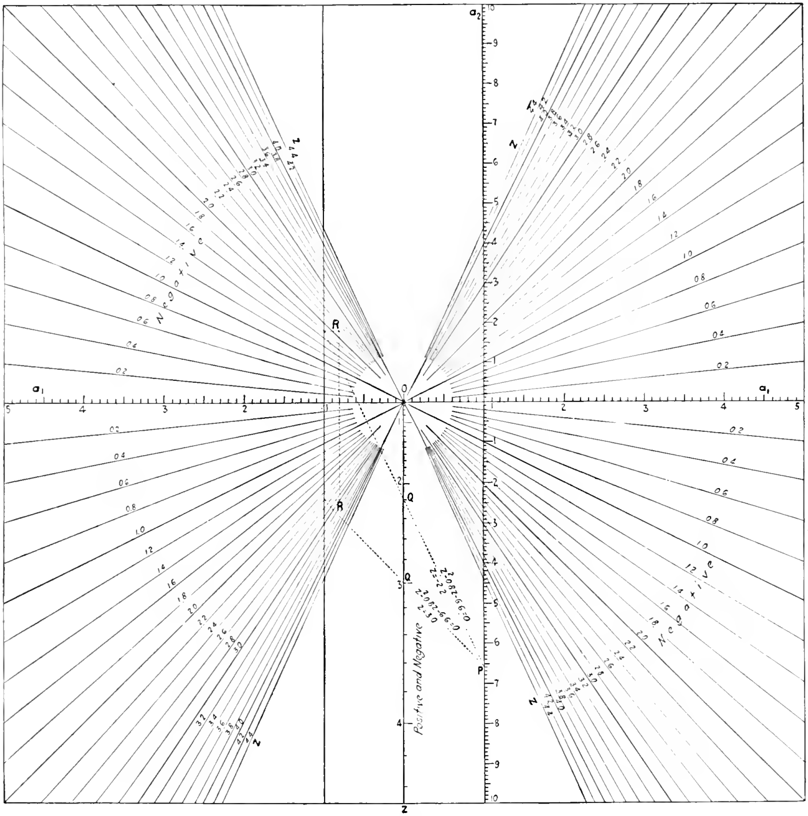


FIG. 77.

For example, suppose that

$$f_{12} = z_1 z_2 \text{ and } g_{12} = \sqrt{z_1 z_2}$$

then it is evident that both variables are eliminated simultaneously from f_{12} and g_{12} and that there results the parabola $y^2 = x$. Whenever both variables are eliminated simultaneously in this way there results always the equation of a curve which is the *curve support for a binary scale.*

In the above example it is seen that to every point of the parabola there corresponds an indefinite number of pairs of values $z_1 z_2$ and to segregate them either one of the defining equations may be used. Choosing $x = z_1 z_2$, any simple family of curves, except the lines parallel to the Y axis, may be selected to define one of the variables, say $y = z_1$; substituting this value of z_1 in the last equation yields

$$x = y z_2$$

and all lines of the two systems which intersect on the *same ordinate* ($x = z_1z_2$) determine pairs of values of z_1 and z_2 which correspond.

However, in using the diagram, the index must always pass through the point P in which the parabola is cut by the ordinate.

It will be noticed that a similar segregation of the variables could have been obtained by starting with

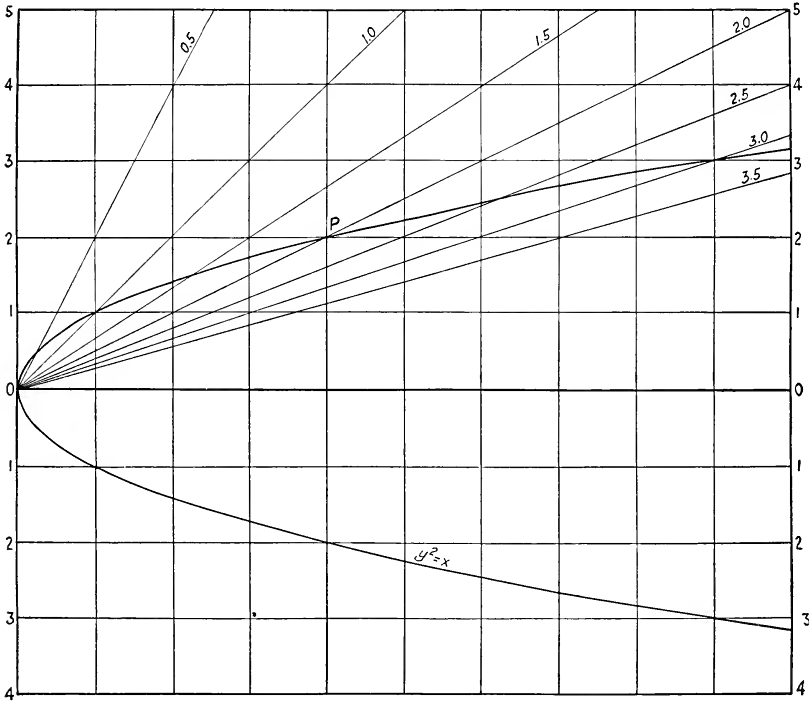


FIG. 78.

the second defining equation $y = \sqrt{z_1z_2}$ and selecting an arbitrary curve net for either z_1 or z_2 and then the corresponding values of z_1 and z_2 would have been found on the curves in the resulting net which intersect on the same abscissa, or line parallel to the axis of X. The scheme of the curved binary scale discussed here is shown in Fig. 78.

24. General Forms of the Equation in Three Variables.—Equations closely allied to Equation (45) will frequently arise. For example, one variable may enter every row of the determinant thus

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{13} & g_{13} & 1 \\ f_1 & g_1 & 1 \end{vmatrix} = 0 \qquad (50)$$

and when z_1 is known no adjustment of the index is required.

Equations (45), (47), (48), (49) and (50) are special cases of the equation of more general form

$$\begin{vmatrix} f_{ij} & g_{ij} & 1 \\ f_{kl} & g_{kl} & 1 \\ f_{mn} & g_{mn} & 1 \end{vmatrix} = 0 \qquad (51)$$

where the subscripts i, j, etc., are allowed to take on in pairs *any* values from the set of numbers 0, 1, 2, 3 with the understanding that 0 shall denote the absence

of a second variable in any function in which it is written. There are eight distinct types of equation (51) if all trivial cases are excluded. These eight types have the following determinants for the left hand member:

$$\overset{\text{I}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_3 & g_3 & 1 \end{vmatrix}}, \quad \overset{\text{II}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_{13} & g_{13} & 1 \end{vmatrix}}, \quad \overset{\text{III}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_{12} & g_{12} & 1 \\ f_{13} & g_{13} & 1 \end{vmatrix}},$$

$$\overset{\text{IV}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_{12} & g_{12} & 1 \\ f_{23} & g_{23} & 1 \end{vmatrix}}, \quad \overset{\text{V}}{\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{13} & g_{13} & 1 \\ f_{23} & g_{23} & 1 \end{vmatrix}}, \quad \overset{\text{VI}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_{23} & g_{23} & 1 \\ f'_{23} & g'_{23} & 1 \end{vmatrix}},$$

$$\overset{\text{VII}}{\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{13} & g_{13} & 1 \\ f'_{13} & g'_{13} & 1 \end{vmatrix}}, \quad \overset{\text{VIII}}{\begin{vmatrix} f_1 & g_1 & 1 \\ f_1 & g_1 & 1 \\ f_{23} & g_{23} & 1 \end{vmatrix}}.$$

It is seen that in cases III and VII it is possible that the unknown variable may enter in each row in which case no initial fixed point would be determined and the position of the index which yields a solution would only be found by trial and error. On the other hand in these same two cases if the variable entering each row is known there is no adjustment of the index required, unless z_3 is unknown in VII. Nor is adjustment required in several other cases, *e.g.*, II when z_3 is unknown, etc. Case I is of course Equation (8) in three variables.

In practice one (or more) of the functions may reduce to a constant in which event a binary scale is needed in carrying out the construction unless the other function in the corresponding row contains but one variable; in that case only a straight scale is involved. If the unknown value is involved in the net of the binary scale as mentioned above the management of the index becomes troublesome unless a known value enters in each row of the determinant.

There are two additional cases in which the number of variables reduces to two, *viz.*:

$$\begin{vmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_{12} & g_{12} & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} f_1 & g_1 & 1 \\ f_{12} & g_{12} & 1 \\ f'_{12} & g'_{12} & 1 \end{vmatrix} = 0$$

and there is still one other permissible case in which all the elements of a row reduce to constants, for example the case

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_3 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

(Compare this equation with Equation (16) Chapter III.) The above equation would be represented by a diagram with a net of curves and a straight scale on the line $y = 1$ and the index would always pass through the origin. No adjustment is needed. If f_{12} and g_{12} should occur in the form $f_1 + g_2$ and $f_1 - g_2$ respectively, ordinary cross-section paper could be utilized to plot the resulting system of perpendicular lines and the scale for the function f_3 would appear on a diagonal line.

25. Equations in More than Three Variables.—It is possible to construct diagrams for equations of the type

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{23} & g_{23} & 1 \\ f_{34} & g_{34} & 1 \end{vmatrix} = 0 \tag{52}$$

In the most general case there will be three nets of curves and it is to be observed that whenever z_1 or z_4 is unknown, no adjustment of the index is necessary. There are many simple cases including the examples of Chapter IV.

Example 51.—The general cubic equation

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0$$

can be written in the reduced determinant form

$$\begin{vmatrix} z^2 & -\dfrac{a_3}{a_2} & 1 \\ 0 & z & 1 \\ -a_2 & -a_1 & 1 \end{vmatrix} = 0$$

The quotient of the two coefficients $\dfrac{a_3}{a_2}$ may be represented by the parameter K and the four variables are then apparent. It is convenient to introduce the scale factor 2 throughout the ordinates and the diagram may be designed on a sheet 20 by 20 inches as shown in Fig. 79.

The defining equations are

$$\begin{aligned} x &= z^2 & \qquad y &= -2K \\ x &= 0 & \qquad y &= 2z \\ x &= -a_2 & \qquad y &= -2a_1 \end{aligned}$$

The diagram must be entered with the values of a_2 and a_1, then the index is rotated about the corresponding point until the value of z on the Y axis is identical with the value found at the point where the index crosses the line $y = -2\frac{a_3}{a_2}$. In the figure the indices are set for the three real roots of the equation

$$z^3 - 0.5\,z^2 - 7.5z + 9 = 0$$

Example 52.—Another excellent example is afforded by the formula for the length of the belt connecting two pulleys and known as the Open Belt Formula

$$L = R(\pi + 2\theta) + r(\pi - 2\theta) + 2C \cos\theta$$

Where R and r are the radii of the two pulleys and C

the distance between their shaft centers. The angle θ is determined by the equation

$$\theta = \arc\sin \frac{R - r}{C}$$

C may be taken 100 as a standard and R and r expressed as decimal parts of C. θ is essentially a parameter. The equation may then be written in the reduced determinant form

$$\begin{vmatrix} 1 & \pi R & 1 \\ -1 & \pi r & 1 \\ \frac{2}{\pi}\theta & \frac{L}{2} - 100\cos\theta & 1 \end{vmatrix} = 0$$

This equation is now a special case of Equation (52) and the defining equations with the horizontal unit ten times the vertical are

$$x = 10 \qquad y = \pi R$$
$$x = -10 \qquad y = \pi r$$
$$x = \frac{20}{\pi}\theta \qquad y = \frac{L}{2} - 100\cos\theta$$

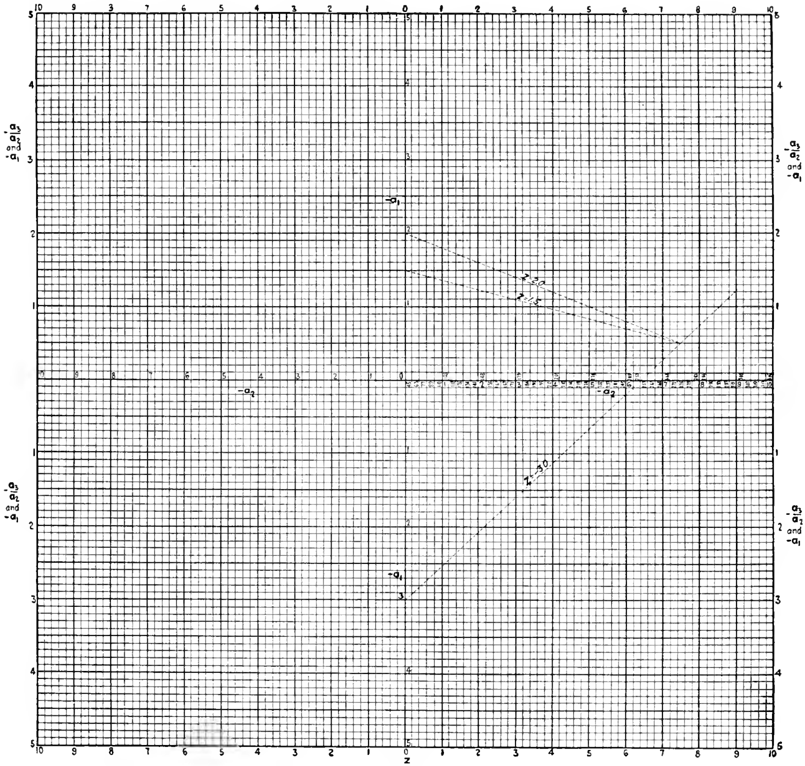


Fig. 79.

Elimination of θ from the last two equations yields the L curves

$$y = \frac{L}{2} - 100\cos\frac{\pi}{20}x$$

Since $(R - r)$ determines θ, the lines parallel to the Y axis are inscribed with the corresponding values of $(R - r)$, Fig. 80.

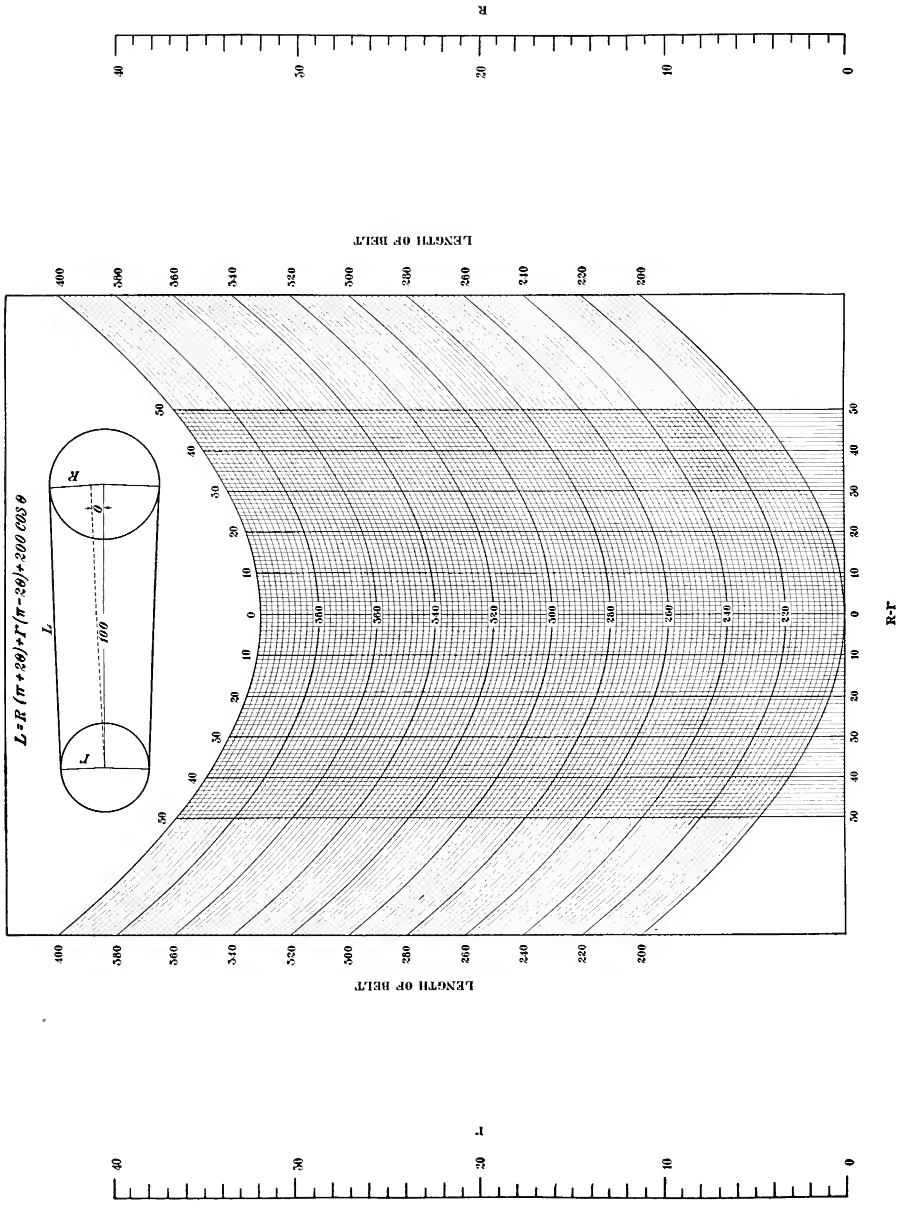

R
LENGTH OF BELT
$L = R(\pi + 2\theta) + r(\pi - 2\theta) + 200 \cos\theta$
L
R
r
100
θ
R-r
LENGTH OF BELT
r

Fig. 80.

The L curves are curves of translation parallel to the Y axis and were drawn with a celluloid template. The diagram is entered with the two values of the pulley radii as decimals of the shaft center distance and the length L of the belt will of course appear as a multiple of the given shaft center distance and will usually have to be changed to inches.

Although no adjustment of the index is necessary to determine the values of L from the diagram of Fig. 80, a related problem for which the same diagram will also serve does require adjustment of the index. This is the problem of finding additional pairs of values of the radii R and r (frequently with a given ratio K) with the same length of belt and shaft distance. To solve this problem it may be assumed that $r = KR$, and the figure is then entered with a tentative value of R and the index turned to the intersection of the given L curve and the vertical line marked with the value $(1 - K)\,R$ or $(R - r)$. Then if the value of r read on the scale for r is found to be less than it should be (*i.e.* KR) it is necessary to select another value of R and repeat the trial and several may be necessary.

It should be noticed that if the parameter K denoting the ratio of the two pulley radii is introduced the determinant equation of the diagram here given has the very special form (L constant)

$$\begin{vmatrix} 1 & \pi R & 1 \\ -1 & \pi KR & 1 \\ f(R) & g(LR) & 1 \end{vmatrix} = 0$$

and the adjustment consists essentially in finding a value of K for which the same value of R results in a collinear position of the index.

Example 53.—The mean temperature difference formula of Professor Greene[1]

$$T = \left[\frac{n(T_1 - T_2)}{T_1^n - T_2^n}\right]^{\frac{1}{1-n}}$$

may be written in the first determinant form

$$\begin{vmatrix} 0 & \dfrac{T^{(1-n)}}{n} & 1 \\ 1 & T_1 & T_1^n \\ 1 & T_2 & T_2^n \end{vmatrix} = 0$$

This formula is used in designing heat transfer apparatus; the coefficient n is dependent on the boundry conditions and varies from not less than 0.08 to 0.50. T is the mean temperature during the elapsed time in which the temperature difference changes from T_1 to a difference of T_2. It is seen at once that a reduced determinant form of the equation obtained by dividing the rows by the elements of the third column would involve the plotting of the reciprocals of the powers of the temperature differences T_1 and T_2 which is to be avoided. If columns one and two were combined for a new third column, then division by its new elements would involve plotting

$$x = \frac{T_i}{1 + T_i} \qquad i = 1, 2$$

which would not give a good disposition of the temperature elements as T varies from 0 to 200. However, by first multiplying column one by 20 there results the reduced determinant form

$$\begin{vmatrix} 1 & \dfrac{n}{T^{(1-n)}} & 1 \\ \dfrac{T_1}{T_1 + 20} & \dfrac{T_1^n}{T_1 + 20} & 1 \\ \dfrac{T_2}{T_2 + 20} & \dfrac{T_2^n}{T_2 + 20} & 1 \end{vmatrix} = 0$$

The exponent n enters each row and as T is the unknown there will be no adjustment of the index. The defining equations are

$$x = 1 \qquad y = \frac{n}{T^{(1-n)}}$$

$$x = \frac{T_1}{T_1 + 20} \qquad y = \frac{T_1^n}{T_1 + 20}$$

$$x = \frac{T_2}{T_2 + 20} \qquad y = \frac{T_2^n}{T_2 + 20}$$

There is a binary scale on the line $x = 1$ but before segregating the variables there involved, an inspection of the two last defining equations shows that the lines

$$x = \frac{T}{T + 20}$$

may be used to advantage; consequently segregate the variables n and T of the binary scale by the equations

$$x = 1 \qquad x = \frac{T}{T + 20} \qquad y = \frac{n(1 - x)^{1-n}}{(20x)^{1-n}}$$

and for the second and third curve nets there follows

$$x = \frac{T_1}{T_1 + 20} \qquad y = \frac{x(1 - x)^{1-n}}{(20x)^{1-n}}$$

$$x = \frac{T_2}{T_2 + 20} \qquad y = \frac{x(1 - x)^{1-n}}{(20x)^{1-n}}$$

The lines parallel to the Y axis are to serve then for the values of the three temperatures T, T_1, T_2. The diagram shown in Fig. 81 was plotted with the horizontal scale unit equal to 20 inches and the vertical scale unit equal 100 inches. It is very suitable for temperature differences up to nearly 100 degrees and will read accurately to one degree. For differences above 100 degrees the readings are less accurate.

There are necessarily two sets of n-curves. The desired value of T is found on the n-curve of the set required in the binary scale and at the intersection of

[1] A. M. Greene, Jr., "Heat Engineering," McGraw-Hill Book Company.

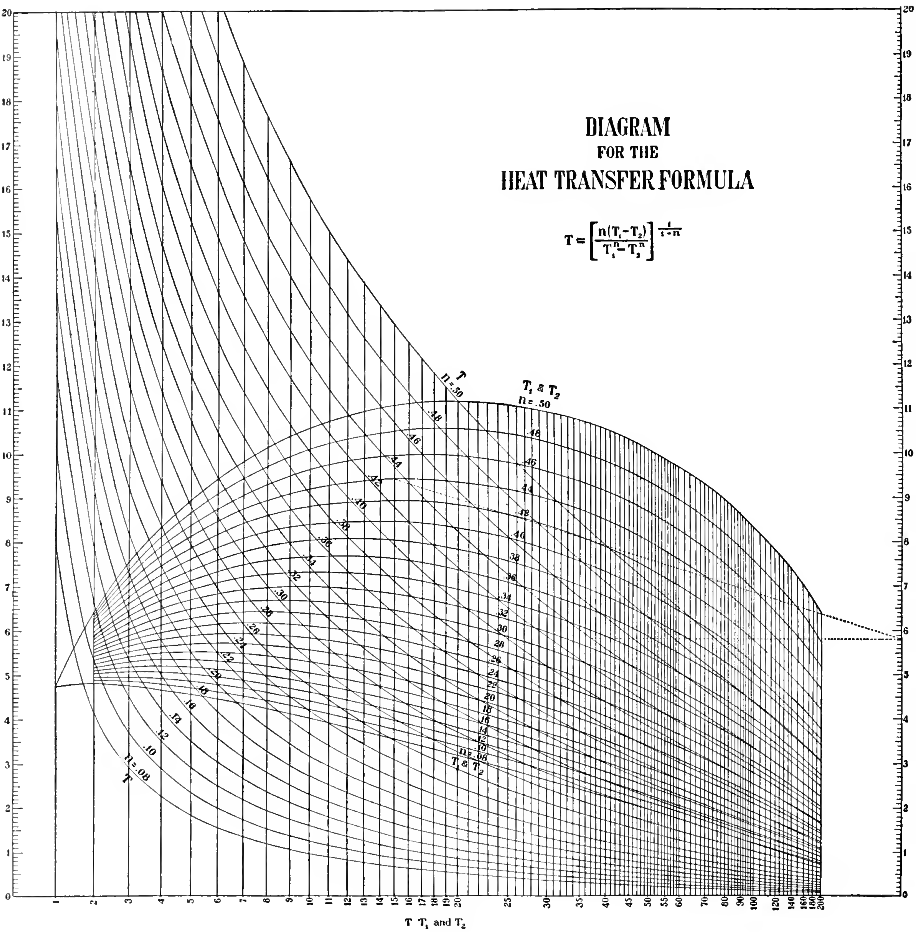


DIAGRAM
FOR THE
HEAT TRANSFER FORMULA
$T = \left[\frac{n(T_1 - T_2)}{T_1^n - T_2^n}\right]^{\frac{1}{1-n}}$
T n = .50
T_1 & T_2 n = .50
n = .08 T
n = .08 T_1 & T_2
T T_1 and T_2

Fig. 81.

a horizontal line drawn from the point of intersection of the index with that scale. The index is of course drawn to join the points corresponding to the given values T_1, n, and T_2, n.

To design a diagram more suited for temperature differences between 100 and 200 degrees it would be desirable to have the space

$$\frac{200}{200+\alpha} - \frac{100}{100+\alpha}$$

on the X axis made a maximum by a suitable choice of α (which was chosen equal to 20 in the drawing here shown). By equating to zero the first derivative with respect to α there results

$$\alpha = \sqrt{20{,}000} = 141.4$$

or more generally; if it is desired to use the diagram primarily for an interval between $T_1 = m$ and $T_2 = n$ it is advantageous to have the interval

$$I = \frac{m}{m+\alpha} - \frac{n}{n+\alpha}$$

a maximum. Equating therefore $\frac{dI}{d\alpha}$ to zero, it is found that

$$\alpha = \sqrt{mn}$$

It is to be observed in the above analysis that there will always be two sets of n-curves in the drawing and that the set which determines T tends to run off the paper with increasing n and to lie near the X axis for small values of n and for large values of α.

Equations in five variables of the form

$$\begin{vmatrix} f_{12} & g_{12} & 1 \\ f_{23} & g_{23} & 1 \\ f_{45} & g_{45} & 1 \end{vmatrix} = 0 \qquad (53)$$

will now present no new difficulties.

Example 54.—Consider the biquadratic equation,

$$z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$$

A first determinant form of the equation is,

$$\begin{vmatrix} z & 1 & 0 \\ -a_2 & a_1 & 1 \\ a_4 + z^4 & -a_3 & z^2 \end{vmatrix} = 0$$

By interchanging rows and columns and dividing the second column by a_2 there results, after the usual modifications, the reduced determinant equation

$$\begin{vmatrix} -z & a_4 + z^4 & 1 \\ \frac{a_2}{a_1} & \frac{a_2}{a_1}a_3 & 1 \\ 0 & -a_2z^2 & 1 \end{vmatrix} = 0$$

This reduced equation is a special case of Equation (53). The defining equations may be written:

$$x = -z \qquad y = \tfrac{1}{5}(a_4 + z^4)$$

$$x = \frac{a_2}{a_1} \qquad y = \tfrac{1}{5}\frac{a_2}{a_1}a_3$$

$$x = 0 \qquad y = -\tfrac{1}{5}a_2z^2$$

The scale factor $\frac{1}{5}$ is needed to restrict the lengths of ordinates in the diagram. The ratio of the coefficients $\frac{a_2}{a_1}$ is regarded as a parameter K. The third pair of equations determines a binary scale on the Y axis. It is convenient to segregate the variables a_2 and z of the binary scale by writing the equations,

$$x = -z \qquad y = -\frac{a_2}{5}x^2$$

The number of curve families is thus reduced since the z lines parallel to the Y axis are made common to the first and third curve nets.

The equations of the first two curve nets are

$$x = -z \qquad y = \tfrac{1}{5}(a_4 + x^4)$$

$$x = \quad K \qquad y = \frac{a_3}{5}x$$

By adopting a modulus of 5 inches there can be shown on a diagram 20 inches square, the numerical value of the roots up to 2. The a_4 curves are quartic curves of translation parallel to the Y axis and in the figure given (Fig. 82) were originally drawn with a celluloid template. The parabolas of the binary scale determine ordinary scales on each ordinate. The diagram is entered with values of the parameter $K = \frac{a_2}{a_1}$ and values of a_3 which determine a point in the second curve net. The index must then be rotated about this point until it intersects the curve marked with the given value of a_4 on the same ordinate that passes through the point on the parabola a_2 cut by the horizontal projecting line from the point of intersection of the index and the Y axis. In Fig. 82 the index is set for the two real roots of the biquadratic equation

$$z^4 + 1.33z^3 + 1.6z^2 + 2z - 3.3 = 0$$

The equation

$$\begin{vmatrix} f_{ij} & g_{ij} & 1 \\ f_{kl} & g_{kl} & 1 \\ f_{mn} & g_{mn} & 1 \end{vmatrix} = 0 \qquad (51)$$

where the subscripts are allowed to take on any two different values in pairs from the numbers 0, 1, 2, 3, 4, and 5, exhausts all possible cases of the equation in five variables.

Problem **1.**—Discuss the equation

$$\begin{vmatrix} 0 & z^2 & 1 \\ 1 & z & 0 \\ -a_1 & -a_2 & 1 \end{vmatrix} = 0$$

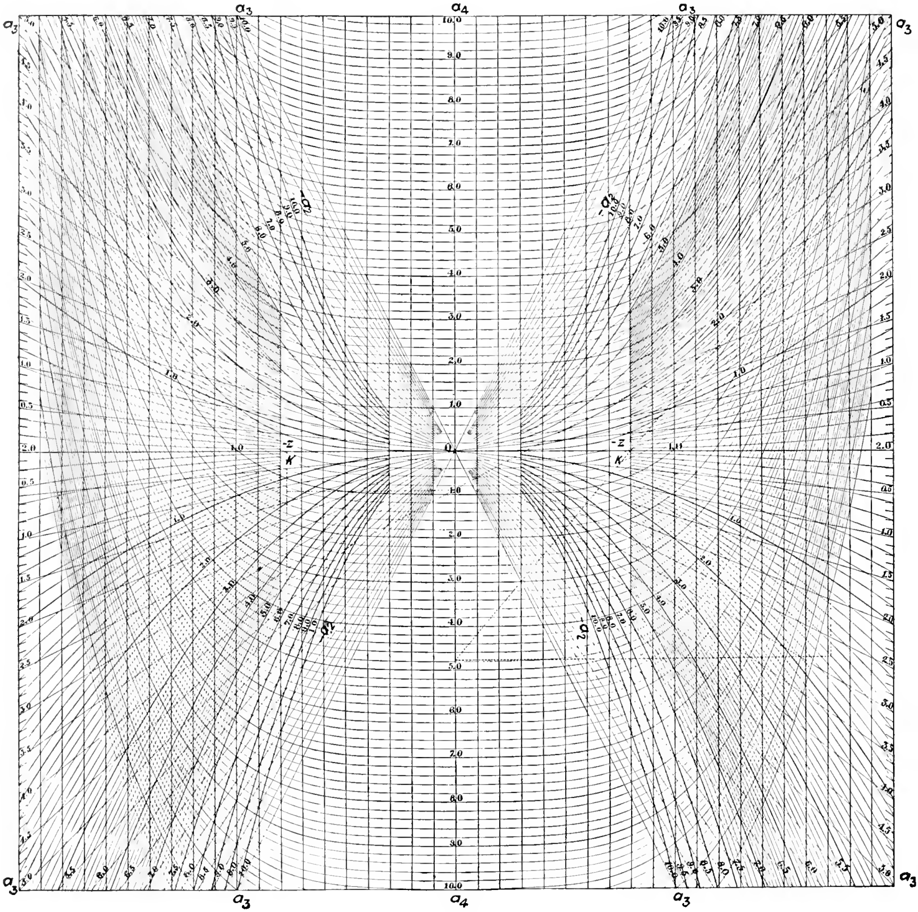


FIG. 82.

Problem 2.—Criticise the diagram for the quadratic equation which results from the determinant form

$$\begin{vmatrix} z^2 + a_1 z & -1 & 1 \\ a_2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Problem 3.—Discuss the case of equation (51) where z_1 is unknown and when the curves for z_1 are the same in two curve nets.

Problem 4.—Discuss the possible diagram for the cubic equation written in the form

$$\begin{vmatrix} 0 & z^2 & 1 \\ 1 & -a_2 z & z \\ 1 & a_3 & -a_1 \end{vmatrix} = 0$$

Problem 5.—The law of cosines in trigonometry may be written in the form

$$\begin{vmatrix} 0 & -\dfrac{a^2}{c} & 1 \\ \dfrac{2b}{2b-1} & \dfrac{b^2+c^2}{c(1-2b)} & 1 \\ 1 & -\cos A & 1 \end{vmatrix} = 0$$

Construct a diagram.

Problem 6.—Construct a diagram for a formula of trigonometry which falls under the special form (Problem 3) of Equation (51).

Problem 7.—The formula

$$\Delta t = \frac{T_1 - T_2}{\log_e \dfrac{T_1}{T_2}}$$

of Fig. 56 is written

$$\Delta t = \frac{T_2 - T_0}{\log_e \dfrac{T_1 - T_0}{T_1 - T_2}}$$

for use in connection with exhaust steam feed water heaters, where

T_1 = temperature of the exhaust steam
T_2 = temperature of the water leaving the heater
T_0 = temperature of the water entering the heater
Δt = average temperature difference.

Show that it can be written in the reduced determinant form

$$\begin{vmatrix} \dfrac{1}{\Delta t} & 0 & 1 \\ \dfrac{\log_e (T_1 - T_2)}{(T_1 - T_2)} & -\dfrac{1}{(T_1 - T_2)} & 1 \\ \dfrac{\log_e (T_1 - T_0)}{(T_1 - T_0)} & -\dfrac{1}{(T_1 - T_0)} & 1 \end{vmatrix} = 0$$

and design a diagram with one curved binary scale which has two systems of segregating curves that serve for the three variables T_1, T_2, T_0.

Problem 8.—If the formula of the preceding problem is written

$$\begin{vmatrix} 0 & \Delta t & 1 \\ -\dfrac{1}{\log_e (T_1 - T_2)} & \dfrac{(T_1 - T_2)}{\log_e (T_1 - T_2)} & 1 \\ -\dfrac{1}{\log_e (T_1 - T_0)} & \dfrac{(T_1 - T_0)}{\log_e (T_1 - T_0)} & 1 \end{vmatrix} = 0$$

design the corresponding diagram.

Problem 9.—The annual sinking fund which will accrue to 1 at the end of n years is given by the formula

$$\frac{1}{S_{\overline{n}|}} = \frac{i}{(1+i)^n - 1}$$

This equation may be given the determinant form

$$\begin{vmatrix} \dfrac{1}{S_{\overline{n}|}} & 0 & 1 \\ \dfrac{i}{(1+i)^n} & \dfrac{1}{(1+i)^n} & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

Identify this with the last special form discussed in this chapter and construct a diagram with suitable scales for practical use for values of n between from 5 to 20 intervals.

Problem 10.—The accumulation of an annuity of 1 per annum at the end of n years is given as the formula $S_{\overline{n}|} = \dfrac{(1+i)^n - 1}{i}$. This equation has a determinant form similar to the one of Problem 9. Construct a useful diagram for va ues of i from 3 to 12 per cent.

APPENDIX A

DETERMINANTS OF THE THIRD ORDER

Definition.—The square array of nine numbers with two vertical bars

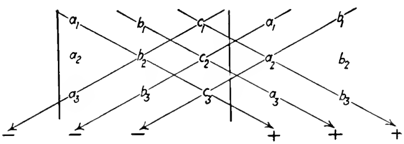
$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is a convenient symbol for the expression,

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3 \quad (1)$$

and is called a *determinant* of the third order. The separate letters are called *elements*. The elements in a vertical line form a *column* and those in a horizontal line a *row*. The expression (1) is called the expansion of the determinant. The elements $a_1b_2c_3$ form the principal diagonal of the determinant and the elements $c_1b_2a_3$ the secondary diagonal.

Expansion or Development of Determinants.—When the determinant Δ is given, the expansion (1) may be obtained as follows: Rewrite the first and second columns to the right of the determinant.



The diagonals running down from left to right give the positive terms. The diagonals running down from right to left give the negative terms. Whenever negative elements are present care must be taken in determining the sign of each term in the expansion.

SIMPLE PROPERTIES OF DETERMINANTS

I. *When all the elements of one row or of one column are zero the value of the determinant is zero.* This is proved by observing that each term in the expansion contains as factors one and only one element from each row and each column.

II. *If all the terms in a row or in a column are multiplied (or divided) by the same number K, the value of the determinant is multiplied (or divided) by K.* The reasoning is the same as for **I.** In particular if $K = -1$ the sign of the determinant is changed.

III. *If the rows of a determinant are changed into corresponding columns the determinant is unchanged.* Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

IV. *If two rows or columns of a determinant are interchanged the sign of the determinant is changed.* This property may be proved for adjacent rows by determining the change in the expansion due to interchange of corresponding subscripts. Repetition of this process will extend the result to any two rows. By virtue of III the result is true for columns.

V. *If a determinant has two rows or columns identical, its value is zero.* If we interchange two rows we obtain by IV $-\Delta$, but since the interchange of identical rows does not alter the determinant we have

$$\Delta = -\Delta$$

that is

$$2\Delta = 0$$

or

$$\Delta = 0$$

VI. *If one row or column of a determinant Δ has as elements the sums of two or more numbers, Δ can be written as the sum of two or more determinants.* Thus

$$\begin{vmatrix} a_1 + a_1' + a_1'' & b_1 & c_1 \\ a_2 + a_2' + a_2'' & b_2 & c_2 \\ a_3 + a_3' + a_3'' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1'' & b_1 & c_1 \\ a_2'' & b_2 & c_2 \\ a_3'' & b_3 & c_3 \end{vmatrix}$$

VII. *The value of any determinant Δ is not changed if each element of any row or column multiplied by any given number K be added to the corresponding element of any other row or column.*

By II and VI

$$\begin{vmatrix} a_1 + Ka_3 & a_2 & a_3 \\ b_1 + Kb_3 & b_2 & b_3 \\ c_1 + Kc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} +$$

$$K \begin{vmatrix} a_3 & a_2 & a_3 \\ b_3 & b_2 & b_3 \\ c_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0$$

Special Properties.—It results, by V immediately from II that

$$\text{if} \qquad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{a_1}{c_1} & \frac{b_1}{c_1} & 1 \\ \frac{a_2}{c_2} & \frac{b_2}{c_2} & 1 \\ \frac{a_3}{c_3} & \frac{b_3}{c_3} & 1 \end{vmatrix} = \frac{\Delta}{c_1c_2c_3} = 0$$

provided that c_1, c_2, c_3 are all different from zero. A column of unit elements may then always be introduced into the equation $\Delta = 0$. For even should a zero appear in every column, by using VII a column of elements all different from zero may be obtained and by using IV, this column may be given the third position. Finally the determinant may be divided by the elements of the third column.

In the construction of engineering diagrams one of the fundamental operations is to write certain given formulas of three variables in the determinant equation form.

$$\begin{vmatrix} \frac{a_1}{c_1} & \frac{b_1}{c_1} & 1 \\ \frac{a_2}{c_2} & \frac{b_2}{c_2} & 1 \\ \frac{a_3}{c_3} & \frac{b_3}{c_3} & 1 \end{vmatrix} = 0$$

Multiplication of Determinants of the Third Order. The product of the two determinants of the third order Δ and Δ_1, is a determinant of the third order as follows:

$$\Delta \cdot \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} m_1 & n_1 & 1 \\ m_2 & n_2 & 2 \\ m_3 & n_3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1m_1 + b_1n_1 + c_11 & a_1m_2 + b_1n_2 + c_12 & a_1m_3 + b_1n_3 + c_13 \\ a_2m_1 + b_2n_1 + c_21 & a_2m_2 + b_2n_2 + c_22 & a_2m_3 + b_2n_3 + c_23 \\ a_3m_1 + b_3n_1 + c_31 & a_3m_2 + b_3n_2 + c_32 & a_3m_3 + b_3n_3 + c_33 \end{vmatrix}$$

To prove this result it will be sufficient to actually carry out the expansions and multiplications. A further proof is given by L. G. Weld "Theory of Determinants," Chapter VI and in any work on determinants.

A working rule for multiplication may then be stated thus: *Connect by plus signs the elements of each row in both determinants. Place the first row of the first determinant upon each row of the second in turn allowing each two elements as they touch to become products. This is the first row of the product. Perform the same operation on the second determinant with the second row of the first to form the second row of the product, and again with the third row of the first determinant to obtain the third row of the product.*

Note that the product (by virtue of III) may also be obtained by using columns instead of rows.

APPENDIX B

THE PROJECTIVE TRANSFORMATION

Definition.—A geometric transformation in the plane is an operation which replaces one geometric configuration by another. A *one to one point transformation* replaces a given point by another uniquely determined point. Under the operation of such a transformation the locus of a given variable point $P(xy)$ is transformed into, or replaced by, another definite locus traced by the corresponding point $P_1(x_1y_1)$.

Equations of a Transformation.—Usually a relation may be written between the coordinates of a given point (xy) and those of the transformed point (x_1y_1). Such equations are called the equations of transformation. Thus for example, if every point P of the plane is pushed outward by an impulse from the origin O so that the distance OP is doubled, there results obviously

$$x_1 = 2x$$
$$y_1 = 2y$$

for the relations connecting the coordinates of the old and the new points. Such a transformation is called

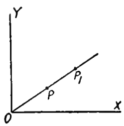


FIG. 83.

a dilatation. By it, circles about the origin are transformed into circles with radii twice as great. Straight lines remain straight, etc. A more general dilatation is given by the equations

$$x_1 = \mu x \qquad y_1 = \mu y$$

where μ is any constant whatever.

Kinds of Point Transformations.—Obviously if a pair of equations

$$x_1 = \phi(xy) \qquad y_1 = \psi(xy) \qquad (1)$$

are written at will, they may in general be regarded as establishing geometrically a relation between the points (xy) and the (transformed) points (x_1y_1) which may be computed whenever values are assigned to x and y; *i.e.*, whenever any point P is selected. Now the properties of the resulting geometric transformation will depend upon the nature of the functions ϕ and ψ in Equation (1). For example if

$$x_1 = x + h$$
$$y_1 = y$$

are the equations, then every point of the plane is moved a distance h parallel to the X axis in the positive direction. A straight line whose equation was

$$Ax + By + C = 0$$

becomes

$$A(x_1 - h) + By_1 + C = 0$$

or

$$Ax_1 + By_1 + (C - Ah) = 0$$

which is obviously another straight line parallel to the first one. The last equation is called the transformed equation and determines the transformed locus. To set up the equation of the transformed locus it is first necessary to solve the equations of the transformation for the variable coordinates x and y in terms of the coordinates x_1 and y_1 of the transformed points and it will be assumed here that this may always be done.

More generally, if the equations of a transformation are

$$\begin{aligned} x_1 &= a_1x + b_1y + c_1 \\ y_1 &= a_2x + b_2y + c_2 \end{aligned} \qquad (2)$$

then a straight line

$$Ax + By + C = 0 \qquad (3)$$

goes into another straight line

$$A_1x_1 + B_1y_1 + C_1 = 0 \qquad (4)$$

For, solving Equations (2) for x and y there results

$$x = \frac{\begin{vmatrix} -c_1 + x_1 & b_1 \\ -c_2 + y_1 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{-b_1y_1 + b_2x_1 - b_2c_1 + b_1c_2}{a_1b_2 - a_2b_1}$$

$$y = \frac{\begin{vmatrix} a_1 & -c_1 + x_1 \\ a_2 & -c_2 + y_1 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{-a_2x_1 + a_1y_1 - a_1c_2 + a_2c_1}{a_1b_2 - a_2b_1}.$$

Substitution of these values of x and y in Equation (3) yields after collecting coefficients, a *linear* Equation (4) in the new variables, x_1y_1 in which Equation A_1, B_1, C_1, are expressions involving a, b, c, only. The reader should now actually make the necessary substitutions and prove that the transformed line under this transformation is always *parallel* to the original line.

The Projective Transformation.—All the transformations whose equations have been written above, have the property that they transform straight lines into straight lines again. They have all been special cases of the general transformation

$$x_1 = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3} \qquad y_1 = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3} \tag{5}$$

which is called the *general projective transformation.* The characteristic of the equations of this transformation is that the functions $\phi(xy)$ and $\psi\,(xy)$ of Equations (1) are linear fractional functions with the *same denominators.* The constant coefficients must satisfy the relation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0^1 \tag{6}$$

otherwise the coefficients are not restricted. The determinant of the inequality (6) is called the *determinant of the transformation.* It is to be observed that the transformation Equations (2) above result from Equations (5) if a_3 and b_3 are chosen equal to zero and c_3 equal to unity

Properties of the Projective Transformation.—There are two principal properties enjoyed by this transformation which are important for the work needed in this book. *First,* the transformation preserves straight lines. Solving Equations (5) for x and y it is found that

$$x = \frac{A_1x_1 + B_1y_1 + C_1}{A_3x_1 + B_3y_1 + C_3} \qquad y = \frac{A_2x_1 + B_2y_3 + C_2}{A_3x_1 + B_3y_1 + C_3}$$

where A, B, C, are expressions involving a, b, c, only and consequently a straight line

$$ax + by + c = 0$$

becomes

$$a(A_1x_1 + B_1y_1 + C_1) + b(A_2x_1 + B_2y_1 + C_2) + c(A_3x_1 + B_3y + C_3) = 0$$

and collecting terms this equation has the final form

$$a'x + b'y + c = 0$$

where $a' = aA_1 + bA_2 + cA_3$, etc., and is consequently the equation of a new or transformed straight line.

Second, the transformation may always be so selected that any four points (no three of which are collinear) may be made to take any four (similarly restricted) positions. This result is accomplished by selecting the suitable coefficients for the equations of the transformation (5). To prove this property of the projective transformation whose equations are written in the form (5) above, assume that the four points given are P_1, P_2, P_3, P_4, with the coordinates $(m_1n_1), (m_2n_2), (m_3n_3), (m_4n_4)$, or more briefly, P_i with coordinates m_in_i, where $i = 1 \;.\;.\;.\; 4$.

Let it be assumed now that these four points are to be transformed into the four new positions whose coordinates are respectively p_iq_i $(i = 1 \;.\;.\;.\; 4)$. There will result immediately from Equations (5) eight necessary relations of the form

$$p_i = \frac{a_1m_i + b_1n_i + c_1}{a_3m_i + b_3n_i + c_3} \qquad q_i = \frac{a_2m_i + b_2n_i + c_2}{a_3m_i + b_3n_i + c_3} \tag{7}$$

which the nine coefficients a, b, c of Equations (5) must satisfy. All these equations will be homogeneous in the quantities a, b, c which are to be found. There is required one more relation or equation to completely determine the nine constants and that relation may be selected at will and of course will be so chosen as to reduce the labor of solving the equations as much as possible.

Example.—Suppose that the four given points are those with the coordinates (0 0), (0 −1), (−1 0), (−1 1) and that it is desired to develop a projective transformation which will transform those four points into the four points (0 0), (0 1), (1 0), (1 1), respectively. Choosing for convenience $c_3 = 1$ the eight equations resulting from Equations (7) upon substitution of these coordinate sets are:

$$\begin{aligned} c_1 &= 0 & c_2 &= 0 \\ 0 &= \frac{-b_1 + c_1}{-b_3 + 1} & 1 &= \frac{-b_2 + c_2}{-b_3 + 1} \\ 1 &= \frac{-a_1 + c_1}{-a_3 + 1} & 0 &= \frac{-a_2 + c_2}{-a_3 + 1} \\ 1 &= \frac{-a_1 - b_1 + c_1}{-a_3 - b_3 + 1} & 1 &= \frac{-a_2 - b_2 + c_2}{-a_3 - b_3 + 1} \end{aligned}$$

This set of equations reduces at once to the set of four linear equations

$$\begin{aligned} -a_1 + a_3 &= 1 \\ -b_2 + b_3 &= 1 \\ -a_1 + a_3 + b_3 &= 1 \\ -b_2 + a_3 + b_3 &= 1 \end{aligned}$$

The solutions are $a_1 = 1$, $a_3 = 0$, $b_2 = -1$, $b_3 = 0$. Consequently the equations of the transformation (5) become

$$x = -x \qquad y = -y$$

The important application of the above principle in the present volume arises in connection with the selection of *scale factors* in the design of the necessary diagrams. Suppose in connection with a nomogram for an equation of three variables z_1, z_2, z_3 it is desirable to

[1] Read ≠ "is different from."

move the z_1 and the z_2 scales from the two parallel straight lines $x = -1$ and $x = 1$ to the two lines $x = -\delta_1$ and $x = \delta_2$ respectively and at the same time to introduce the scale factors μ_1 and μ_2 so that the two parallel scales will then have the defining equations

$$x = -\delta_1 \qquad y = \mu_1 g_1$$
$$x = \delta_2 \qquad y = \mu_2 g_2$$

respectively. In order to determine once for all what will be the nature of the change in the defining equations for the third scale it is only necessary to observe that the change determined by the choice of the two transformed scales above is sufficient to determine a projective transformation. The four points (1 0), (1 1), (−1 0), (−1 1), have been transformed respectively into the four points $(\delta_2 0), (\delta_2\mu_2), (-\delta_1 0), (-\delta_2\mu_1)$. Following the procedure above there result the eight equations

$$\delta_2 = \frac{a_1 + c_1}{a_3 + c_3} \qquad 0 = \frac{a_2 + c_2}{a_3 + c_3}$$
$$\delta_2 = \frac{a_1 + b_1 + c_1}{a_3 + b_{33} + c} \qquad \mu_2 = \frac{a_2 + b_2 + c_2}{a_3 + b_3 + c_3}$$
$$-\delta_1 = \frac{-a_1 + c_1}{-a_3 + c_3} \qquad 0 = \frac{-a_2 + c_2}{-a_3 + c_3}$$
$$-\delta_1 = \frac{-a_1 + b_1 + c_1}{-a_3 + b_3 + c_3} \qquad \mu_1 = \frac{-a_2 + b_2 + c_2}{-a_3 + b_3 + c_3}$$

Selecting for convenience the arbitrarily chosen relation

$$a_3 + b_3 + c_3 = 1$$

the solution of the nine equations yields

$$a_1 = \frac{\mu_1\delta_2 + \mu_2\delta_1}{2\mu_1} \qquad b_1 = 0 \qquad c_1 = \frac{\mu_1\delta_2 - \mu_2\delta_1}{2\mu_2}$$
$$a_2 = 0 \qquad b_2 = \mu_2 \qquad c_2 = 0$$
$$a_3 = \frac{\mu_1 - \mu_2}{2\mu_1} \qquad b_3 = 0 \qquad c_3 = \frac{\mu_1 + \mu_2}{2\mu_1}$$

and by substituting these values in the Equations (5) above there results for the necessary projective transformation

$$x = \frac{(\mu_1\delta_2 + \mu_2\delta_1)x + (\mu_1\delta_2 - \mu_2\delta_1)}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$
$$y = \frac{2\mu_1\mu_2 y}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$

There is a convenient modification of these equations if

$$(\mu_1\delta_2 - \mu_2\delta_1) = 0$$

and also another convenient simplification if $\delta_1 = \delta_2 = \delta$ and $(\mu_1\delta_1 - \mu_2\delta_2) \neq 0$ from which results

$$x_1 = \delta\frac{(\mu_1 + \mu_2)x + (\mu_1 - \mu_2)}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$
$$y_1 = \frac{2\mu_1\mu_2 y}{(\mu_1 - \mu_2)x + (\mu_1 + \mu_2)}$$

These are the equations of Chapter III numbered (26).

The equations developed for the introduction of scale factors into the equations numbered (10) and (13) in Chapter III may be obtained by the method here used.

In the text of the present volume the *supplementary* transformations that have been introduced to better the design of diagrams are all very simple and similar transformations can usually be selected by inspection; it is desirable to point out, however, that in the design of important nomograms for permanent service the use of the four point method here described may be the only way that the necessary transformation can be determined.

It is obvious from Equations (5) that if a point P with the coordinates $(m\ n)$ is to be transformed to infinity it is only necessary to choose $a_3b_3c_3$ so that $a_3m + b_3n\mu + c_3 = 0$, since then the values of both x_1 and y_1 will be infinite. The equations of transformation numbered (2) above are the most general equations for the projective transformation which preserves parallelism of straight lines. Such projective transformations are called *affine transformations.*

The Projective Transformation and Determinants. The condition that three points $(x'y')$, $(x''y'')$, $(x'''y''')$ shall lie on a straight line is conveniently expressed in the form

$$\begin{vmatrix} x' & y' & 1 \\ x'' & y'' & 1 \\ x''' & y''' & 1 \end{vmatrix} = 0$$

If a general projective transformation is applied to all the points in the plane the three points in question go over into three new points which are collinear also. Substituting for x' and y', etc., in the above determinant the corresponding values obtained above from Equations (5) in terms of x'_1 and y'_1, etc., there results

$$\begin{vmatrix} \dfrac{A_1x_1' + B_1y_1' + C_1}{A_3x_1' + B_3y_1' + C_3} & \dfrac{A_2x_1' + B_2y_1' + C_2}{A_3x_1' + B_3y_1' + C_3} & 1 \\ \dfrac{A_1x_1'' + B_1y_1'' + C_1}{A_3x_1'' + B_3y_1'' + C_3} & \dfrac{A_2x_1'' + B_2y_1'' + C_2}{A_3x_1'' + B_3y_1'' + C_3} & 1 \\ \dfrac{A_1x_1''' + B_1y_1''' + C_1}{A_3x_{11}''' + B_3y_1''' + C_3} & \dfrac{A_2x_1''' + B_2y_1''' + C_2}{A_3x_1''' + B_3y_1''' + C_3} & 1 \end{vmatrix} = 0$$

and multiplying this equation by the three denominators of the elements of the first column, there results

$$\begin{vmatrix} A_1x' + B_1y' + C_1 & A_2x' + B_2y' + C_2 & A_3x' + B_3y' + C_3 \\ A_1x'' + B_1y'' + C_1 & A_2x'' + B_2y'' + C_2 & A_3x'' + B_3y'' + C_3 \\ A_1x''' + B_1y''' + C_1 & A_2x''' + B_2y''' + C_2 & A_3x''' + B_3y''' + C_3 \end{vmatrix} = 0$$

which by the multiplication law of determinants is

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \times \begin{vmatrix} x_1' & y_1' & 1 \\ x_1'' & y_1'' & 1 \\ x_1''' & y_1''' & 1 \end{vmatrix} = 0$$

Since now the first determinant factor does not vanish[1] the second must and hence the condition that the transformed points lie also upon a straight line appears at once as a result of their original collinear position.

If now it is desired to write the above equation in terms of the original coordinates there follows:

$$\begin{vmatrix} x_1' & y_1' & 1 \\ x_1'' & y_1'' & 1 \\ x_1''' & y_1''' & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x' & y' & 1 \\ x'' & y'' & 1 \\ x''' & y''' & 1 \end{vmatrix} = 0$$

Which may be proved by the laws of multiplication of determinants and Equations (5).

There results then the *Working Rule:*

To apply a projective transformation to the variable elements of a determinant multiply the determinant by the determinant of the transformation. This rule may be used as a check in the practice involved in this volume. The important principle, however, which the above rule brings out is in connection with the manipulation of first determinant equations to reduce them: Every manipulation of a determinant equation by the laws of determinants is equivalent to applying to its elements a projective transformation. In other words every change in the first determinant form has corresponding to it a geometric change in the plane.

[1] This can be shown to be true from the condition VI.

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